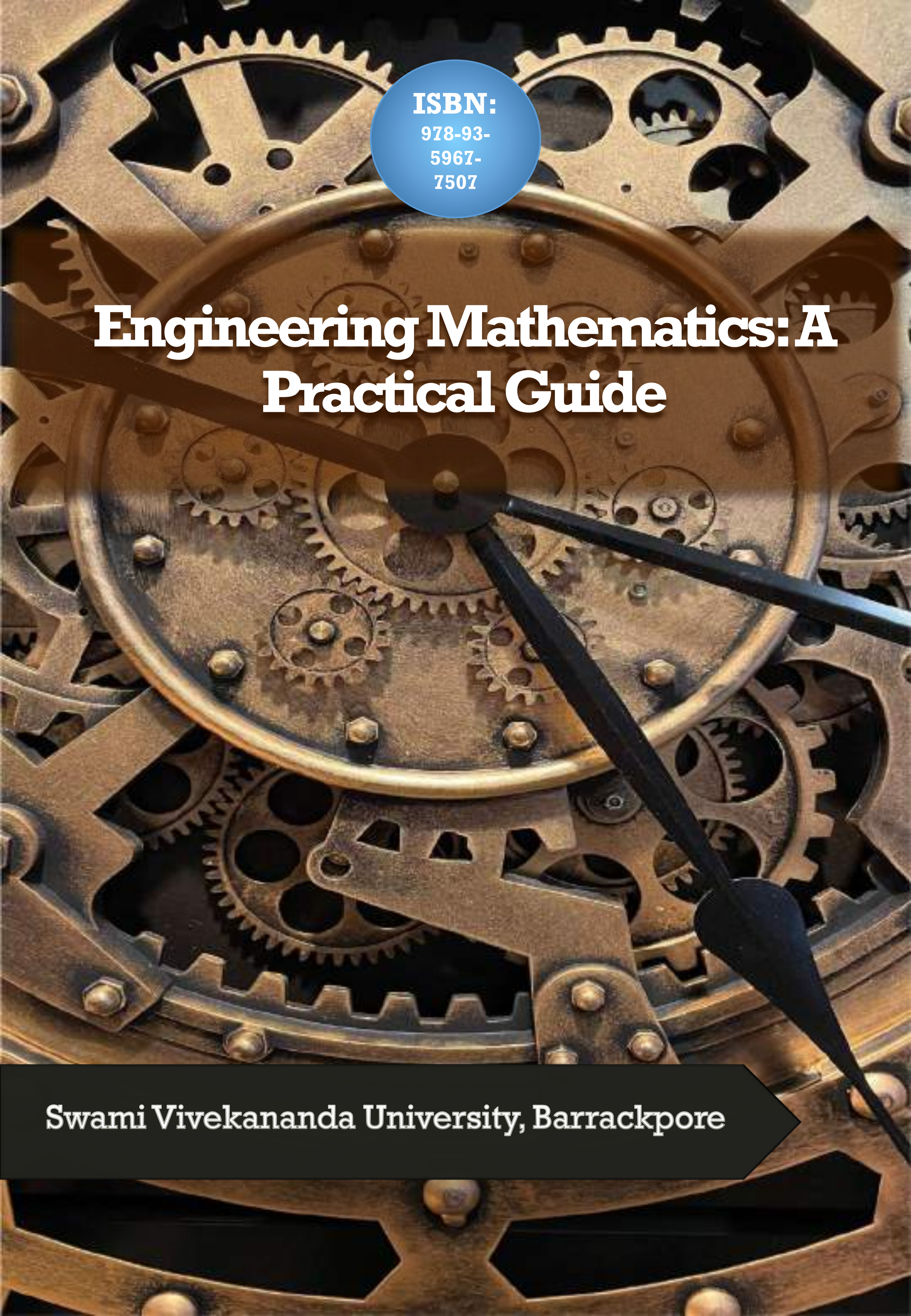

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Engineering Mathematics: A Practical Guide

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Preface

This book covers engineering mathematics in a comprehensive, complete, and up-to-date manner. It is designed to teach engineering, physics, mathematics, computer science, and allied students to the areas of applied mathematics that are most useful for solving real-world problems. This work will give students a foundation in mathematical concepts, allowing them to solve mathematical, scientific, and related engineering problems. Furthermore, the content will include engineering applications and mathematical concepts required for progression into a variety of Incorporated engineering degree programs. It is commonly acknowledged that a student's ability to apply mathematics is a critical factor in determining future success. The content is divided into the following five sections:

1. Functions of Several Variables
2. Matrices
3. Ordinary Differential Equations
4. Partial Differential Equations
5. Laplace Transforms

Furthermore, every chapter is maintained as autonomously as possible (If necessary, any requirements for understanding the specific content of earlier chapters are spelled out at the beginning of each one). We provide the instructor with the greatest amount of freedom to choose the materials and modify them to suit their needs. The current state of engineering mathematics is largely due to the contributions made by the book. With a contemporary approach to the topics mentioned above, this book will get students ready for both the responsibilities of today and the future. We give students the resources and learning aids they need to build a solid foundation in engineering mathematics, which will benefit them in both their future academic endeavours and professional endeavours. General Features of the Book Include:

- Examples are kept simple to make the book easily teaches.
- Independence of chapters and sections to provide for freedom in customizing courses to meet individual needs.
- Presentation is self-contained, with the exception of a few well-defined instances when a proof would go beyond the bounds of the book and a citation is provided in its place.
- A smooth transition from easy to harder content to guarantee a positive teaching and learning environment.
- To assist students in mathematics, engineering, statistics, physics, computer science, and other fields, contemporary standard notation is available in books, journals, and other courses. In addition, we aimed to create a book that would serve as a comprehensive, authoritative, and easily accessible resource for learning and teaching applied mathematics. This would remove the need for laborious online searches or lengthy journeys to the library to get specific reference materials.

Further comments and suggestions for improving the book will be gratefully received.

(Dr. Subhabrata Mondal) 05-12-23

Assistant Professor, Swami Vivekananda University,

Kolkata, West Bengal, India

Acknowledgement

I am writing to express my appreciation to Swami Vivekananda University in Kolkata, India, for all of their help and encouragement in producing this book, "Engineering Mathematics: A Practical Guide" The university's dedication to supporting research and teaching has been important in determining the focus and substance of this publication. We really appreciate collaborative environment and resources of Swami Vivekananda University, Kolkata, which have made it possible for us to research and disseminate the newest developments in a variety of sectors. We hope that this book, which reflects our shared commitment to knowledge, advancement, and the pursuit of quality, will prove to be a useful tool for this prestigious institution as well as the larger academic community.

With sincere appreciation,

(Dr. Subhabrata Mondal) 05-12-23

Assistant Professor, Swami Vivekananda University,

Kolkata, West Bengal, India

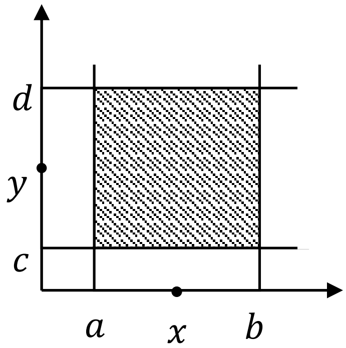
CONTENTS

Chapter 1
Functions of Several Variables
1.1 Sets in n-dimensional Real Space

The set $\mathbb{R}^n$ is defined by $\{(x_1, x_2, \dots, x_n): x_1, x_2, \dots, x_n \in \mathbb{R}\}$. In particular, $\mathbb{R}^2$ is the set of all order pair $\{(x, y): x, y \in \mathbb{R}\}$. Thus a point X in $\mathbb{R}^2$ is defined by $X = (x, y)$ where $x, y \in \mathbb{R}$. Similarly a point X in $\mathbb{R}^3$ is defined by $X = (x, y, z)$ where $x, y, z \in \mathbb{R}$.

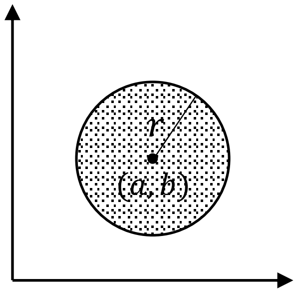
1.1 Cell in $\mathbb{R}^2$

Let $a, b \in \mathbb{R}$ with $a < b$ and $c, d \in \mathbb{R}$ with $c < d$. The set of all ordered pair $\{(x, y) \in \mathbb{R}^2: a < x < b, c < y < d\}$ is said to be an ***open cell*** in $\mathbb{R}^2$ and the set of all ordered pair $\{(x, y) \in \mathbb{R}^2: a \le x \le b, c \le y \le d\}$ is said to be a ***closed cell*** in $\mathbb{R}^2$.



1.2 Disc in $\mathbb{R}^2$

Let $a, b \in \mathbb{R}$ and $r > 0$. The set of all ordered pair $\{(x, y) \in \mathbb{R}^2: (x - a)^2 + (y - b)^2 < r^2\}$ is said to be an ***open disc*** in $\mathbb{R}^2$ about (a, b) and the set of all ordered pair $\{(x, y) \in \mathbb{R}^2: (x - a)^2 + (y - b)^2 \le r^2\}$ is said to be a ***closed disc*** in $\mathbb{R}^2$ about (a, b).



1.3 Open ball in $\mathbb{R}^3$

Let $a, b, c \in \mathbb{R}$ and $r > 0$. The set of all ordered pair $\{(x, y, z) \in \mathbb{R}^3: (x - a)^2 + (y - b)^2 + (\mathrm{z} - \mathrm{c})^2 < r^2\}$ is said to be an ***open ball*** in $\mathbb{R}^3$ about (a, b, c).

1.4 Neighbourhood of a point

Let $(a, b, c) \in \mathbb{R}$ (or $(a, b) \in \mathbb{R}^2$). A set $S \subseteq \mathbb{R}^3$ (or $\mathbb{R}^2$) is said to be a ***neighbourhood*** of (a, b, c) (or (a, b)) if there exists an open ball (or open disk) about (a, b, c) (or (a, b)) such that the open ball (or open disk) is a subset of S.

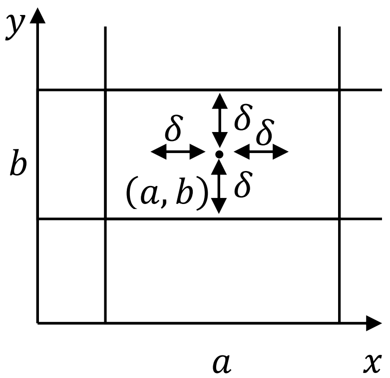
- **Note:**

(i) An open disk about $(a,b) \in \mathbb{R}^2$ is also a neighbourhood of (a,b).

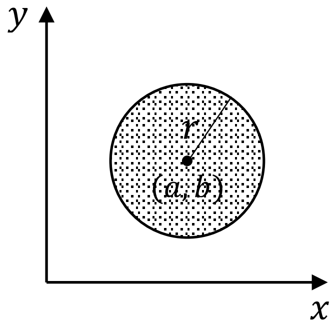
(ii) An open cell about (a,b) is a rectangular neighbourhood of (a,b), i.e. $\{(x,y) \in \mathbb{R}^2 : |x-a| < \delta, |y-b| < \delta\}$ where $\delta > 0$ is a neighbourhood of (a,b).

(iii) $\{(x,y) \in \mathbb{R}^2 : 0 < (x-a)^2 + (y-b)^2 \leq r^2\}$ and $\{(x,y) \in \mathbb{R}^2 : 0 < |x-a| < \delta, 0 < |y-b| < \delta\}$ where $\delta > 0$, are deleted neighbourhoods of (a,b).

(iv) An open ball $(a,b,c) \in \mathbb{R}^3$ is also an neighbourhood of (a,b).



Rectangular neighbourhood



Circular neighbourhood

1.5 Interior point

Let $S \subseteq \mathbb{R}^3$. A point $(a,b,c) \in S$ is said to be an ***interior point*** of S if there exists a neighbourhood N of (a,b,c) such that $N \subseteq S$. The set of all interior points of S is said to be the ***interior*** of S and is denoted by ***int S***.

1.6 Open set

A set $S \subseteq \mathbb{R}^3$ is said to be an ***open set*** in $\mathbb{R}^3$ if each point of S is an interior point of S. Clearly $\mathbb{R}^3$ and ϕ both are open sets in $\mathbb{R}^3$.

1.7 Worked example

1. Show that $S = \{(x,y) : -1 < x < 1, -2 < y < 2\}$ is an open set.

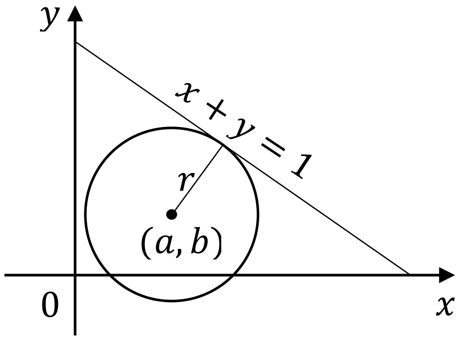
→ Let $(a,b) \in S$. Then S is itself a neighbourhood of (a,b). Therefore (a,b) is an interior point of S. Since (a,b) is an arbitrary point of S, therefore each point in S is an interior point of S. Hence S is an open set.

2. **Show that $S = \{(x, y): (x-3)^2 + (y-5)^2 = 100\}$ is an open set.**

→ Let $(a, b) \in S$. Then S is itself a neighbourhood of (a, b). Therefore (a, b) is an interior point of S. Since (a, b) is an arbitrary point of S, therefore each point in S is an interior point of S. Hence S is an open set.

3. **Justify your answer: $S = \{(x, y): x + y < 1\}$ is an open set.**

→ Let $(a, b) \in S$ be any point and r be the perpendicular distance between (a, b) and the straight line $x + y = 1$.



We define a neighbourhood N of (a, b) by

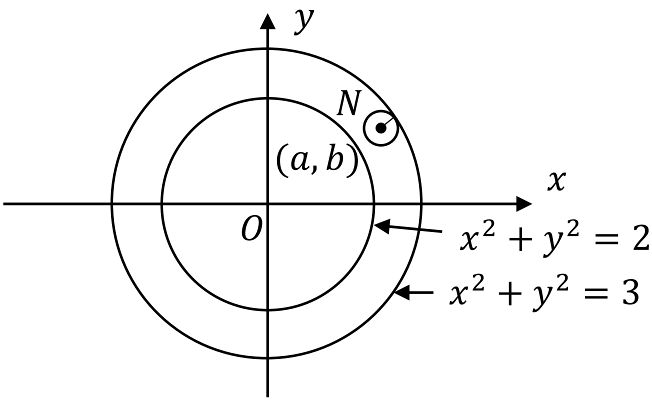
$$N = \{(x, y) \in \mathbb{R}^2: (x-a)^2 + (y-b)^2 < r^2\}$$

Clearly $N \subseteq S$. Therefore (a, b) is an interior point of S. Since (a, b) is an arbitrary point of S, therefore each point in S is an interior point of S. Hence S is an open set.

4. **Is $S = \{(x, y): 2 < x^2 + y^2 < 3\}$ an open set?**

→ Let $(a, b) \in S$ be any point and r_1, r_2 be the shortest distance between (a, b) and the circles $x^2 + y^2 = 2$ and $x^2 + y^2 = 3$.

Let $r = \min\{r_1, r_2\}$.



We define a neighbourhood N of (a, b) by

$$N = \{(x, y) \in \mathbb{R}^2: (x-a)^2 + (y-b)^2 < r^2\}$$

Clearly $N \subseteq S$.

Therefore (a, b) is an interior point of S. Since (a, b) is an arbitrary point of S, therefore each point in S is an interior point of S. Hence S is an open set.

5. **Is $S = \{(x, y): x^2 < y\}$ an open set?**

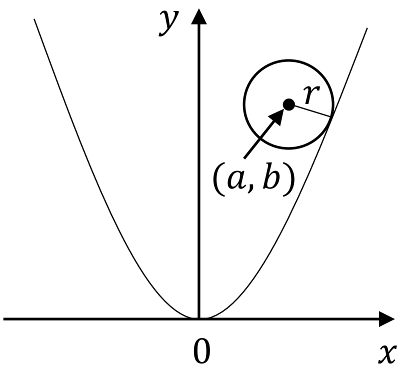
→ Let $(a, b) \in S$ be any point and r be the shortest distance between (a, b) and the parabola $x^2 = y$.

We define a neighbourhood N of (a,b) by

$N = \{(x,y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 < r^2\}$

Clearly $N \subseteq S$.

Therefore (a,b) is an interior point of S. Since (a,b) is an arbitrary point of S, therefore each point in S is an interior point of S. Hence S is an open set.



- **Note:** Similarly the following sets are open sets in $\mathbb{R}^2$:

(i) $S = \{(x,y) \in \mathbb{R}^2 : 4x^2 + 9y^2 < 36\}$

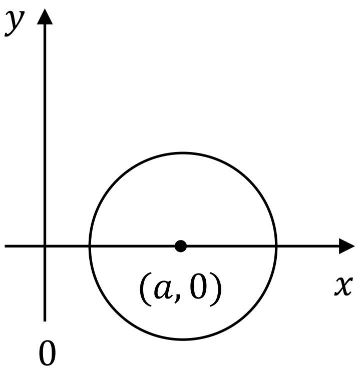
(ii) $S = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$

(iii) $S = \{(x,y) \in \mathbb{R}^2 : xy < 1\}$

(iv) $S = \{(x,y) \in \mathbb{R}^2 : x < y\}$

6. Determine whether $\boldsymbol{S = \{(x,0) : x \in \mathbb{R}\}}$ an open set.

→ Let $(a,0) \in S$ be any point. Then every neighbourhood of $(a,0)$ contains infinitely many points which do not belong to S. Therefore $(a,0)$ is not an interior point of S. Hence S is not an interior point of S.



- **Note:** Similarly $S = \left\{\left(\frac{1}{m}, \frac{1}{n}\right) \in \mathbb{R}^2 : m,n \in \mathbb{N}\right\}$ is not an open set in $\mathbb{R}^2$.

➢ **Theorem 1.7.1: The union of a finite number of open sets in $\mathbb{R}^3$ is an open set in $\mathbb{R}^3$.**

Proof: Let $S_1, S_2, \dots, S_m$ be m open sets in $\mathbb{R}^3$ and

$$S = \bigcup_{i=1}^{m} S_i$$

Let $(a,b,c) \in S$. Then $(a,b,c) \in S_i$ for some $i = 1,2,\dots,m$. Since $(a,b,c) \in S_i$ and S_i is an open set in $\mathbb{R}^3$, (a,b,c) is an interior point of S_i.

Therefore there exists a nbd N of (a,b,c) such that $N \subseteq S_i$. Since $S_i \subseteq S$, therefore $N \subseteq S$.

This shows that (a,b,c) is also an interior point of S. Since (a,b,c) is an arbitrary point of S, therefore each point in S is an interior point of S. Hence S is an open set.

This completes the proof.

- **Theorem 1.7.2: The union of arbitrary collections of open sets in $\mathbb{R}^3$ is an open set in $\mathbb{R}^3$.**

Proof: Let $\{S_i : i \in \Lambda\}$, Λ being a subset of $\mathbb{N}$, be a collection of open sets in $\mathbb{R}^3$ and

$$S = \bigcup_{i=\Lambda} S_i$$

Let $(a,b,c) \in S$. Then $(a,b,c) \in S_i$ for some $i \in \Lambda$. Since $(a,b,c) \in S_i$ and S_i is an open set in $\mathbb{R}^3$, (a,b,c) is an interior point of S_i. Therefore there exists a nbd N of (a,b,c) such that $N \subseteq S_i$. Since $S_i \subseteq S$, therefore $N \subseteq S$.

This shows that (a,b,c) is also an interior point of S. Since (a,b,c) is an arbitrary point of S, therefore each point in S is an interior point of S. Hence S is an open set.

- **Theorem 1.7.3: The intersection of finite number of open sets in $\mathbb{R}^3$ is an open set in $\mathbb{R}^3$.**

Proof: Let $S_1, S_2, \dots, S_m$ be m open sets in $\mathbb{R}^3$ and

$$S = \bigcap_{i=1}^{m} S_i$$

Two cases arise.

Case-1: $S = \phi$

Then S is an open set in $\mathbb{R}^3$.

Case-2: $S \neq \phi$

Let $(a,b,c) \in S$. Then $(a,b,c) \in S_i$ for each $i = 1,2,\dots,m$. Since $(a,b,c) \in S_i$ and S_i is an open set in $\mathbb{R}^3$, (a,b,c) is an interior point of S_i. Therefore there exists $r_i > 0$ such that

$$N_i = \{(x,y,z) \in \mathbb{R}^3 : (x-a)^2 + (y-b)^2 + (z-c)^2 \le r_i^2\} \subseteq S_i$$

Let $r = \min\{r_1, r_2, \dots, r_m\}$. Then

$$N = \{(x,y,z) \in \mathbb{R}^3 : (x-a)^2 + (y-b)^2 + (z-c)^2 \le r^2\} \subseteq N_i$$

for each $i = 1,2,\dots,m$.

Thus we have

$$N \subseteq S_i \qquad \text{for each} \quad i = 1,2,\dots,m$$

$$\Longrightarrow N \subseteq \bigcap_{i=1}^{m} S_i = S$$

This shows that (a,b,c) is an interior point of S. Since (a,b,c) is an arbitrary point of S, therefore each point in S is an interior point of S. Hence S is an open set.

- **Note:** The intersection of an infinite number of open sets in $\mathbb{R}^3$ may be or may not be an open set in $\mathbb{R}^3$.

Let us consider the collection of open sets

$$S_n = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < \frac{1}{n^2}\right\}, n \in \mathbb{N}$$

Then

$$S = \bigcap_{n=1}^{\infty} S_n = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 0\} = \{(0,0,0)\}$$

which is not an open set in $\mathbb{R}^3$.

Next we consider the collection of open sets

$$S_n = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < n\}, n \in \mathbb{N}$$

Then

$$S = \bigcap_{n=1}^{\infty} S_n = S_1$$

which is an open set in $\mathbb{R}^3$.

1.8 Closed set

A set $S \subseteq \mathbb{R}^3$ is said to be a ***closed set*** in $\mathbb{R}^3$ if the complement of S in $\mathbb{R}^3$ is an open set in $\mathbb{R}^3$. Clearly $\mathbb{R}^3$ and ϕ both are closed sets in $\mathbb{R}^3$.

1.8.1 Examples

1. $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \geq r^2\}$ is a closed set in $\mathbb{R}^3$, since $S^c = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < r^2\}$ is an open set in $\mathbb{R}^3$.
2. $\mathbb{R}^3$ and ϕ are closed sets in $\mathbb{R}^3$.

➢ **Theorem 1.8.1: The union of finite number of closed sets in $\mathbb{R}^3$ is closed**.

Proof: Let $F_1, F_2, \dots, F_m$ be m closed sets in $\mathbb{R}^3$ and

$$F = \bigcup_{i=1}^{m} F_i$$

Now

$$F^c = \bigcap_{i=1}^{m} F_i^c$$

Since each F_i^c is an open set in $\mathbb{R}^3$ and intersection of finite number of open sets is also an open set in $\mathbb{R}^3$, therefore F^c is an open set in $\mathbb{R}^3$. Hence F is an closed set in $\mathbb{R}^3$.

This completes the proof.

- **Note:** Similarly the intersection of finite number of closed sets in $\mathbb{R}^3$ is closed in $\mathbb{R}^3$.

➢ **Theorem 1.8.2: The intersection of arbitrary number of closed sets in $\mathbb{R}^3$ is closed in $\mathbb{R}^3$.**

Proof: Let $\{F_i : i \in \Lambda\}$, Λ being a subset of $\mathbb{N}$, be a collection of closed sets in $\mathbb{R}^3$ and

$$F = \bigcap_{i=\Lambda} F_i$$

Now

$$F^c = \bigcup_{i=\Lambda} F_i^c$$

Since each F_i^c is an open set in $\mathbb{R}^3$ and union of arbitrary collection of open sets is also an open set in $\mathbb{R}^3$, therefore F^c is an open set in $\mathbb{R}^3$. Hence F is an closed set in $\mathbb{R}^3$.

- **Note:** The union of an infinite number of closed sets in $\mathbb{R}^3$ may or may not be a closed set in $\mathbb{R}^3$.

Let us consider the collection of closed sets

$$F_n = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 1 - \frac{1}{n}\right\}, n \in \mathbb{N}$$

Then

$$F = \bigcup_{i=1}^{\infty} F_i = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$$

which is not a closed set in $\mathbb{R}^3$.

Next we consider the collection of closed sets

$$F_n = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le \frac{1}{n}\right\}, n \in \mathbb{N}$$

Then

$$F = \bigcup_{i=1}^{\infty} F_i = F_1$$

which is a closed set in $\mathbb{R}^3$.

1.8.2 Worked example

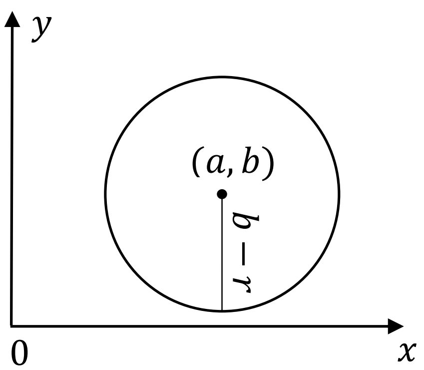
1. Show that $F = \{(a,0) : a \in \mathbb{R}\}$ is a closed set in $\mathbb{R}^2$.

→ Here $F^c = \{(x,y) \in \mathbb{R}^2 : y \ne 0\} = F_1 \cup F_2$ where

$$F_1 = \{(x,y) \in \mathbb{R}^2 : y > 0\} \quad \text{and} \quad F_2 = \{(x,y) \in \mathbb{R}^2 : y < 0\}$$

Let $(a,b) \in F_1$. Then $b > 0$. By Archimedean property, there exists a real number r such that $0 < r < b$. We now define a nbd N of (a,b) as

$$N = \{(x,y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 \le (b-r)^2\}$$



Clearly $N \subseteq F_1$. Therefore (a,b) is an interior point of F_1. Since (a,b) is arbitrary, therefore each point in F_1 is an interior point of F_1. Hence F_1 is an open set.

Similarly we can show that F_2 is an open set.

Since finite union of open sets is an open set, therefore $F^c = F_1 \cup F_2$ is an open set in $\mathbb{R}^2$.

Consequently, F is a closed set in $\mathbb{R}^2$.

- **Note:** Similarly the set $F = \{(x, y) \in \mathbb{R}^2: 1 \leq x^2 + y^2 \leq 4\}$ is a closed set in $\mathbb{R}^2$.

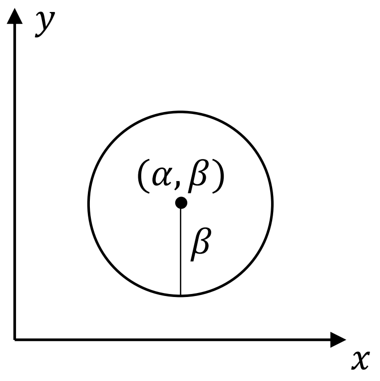
1.9 Limit point

A point (a, b, c) in $\mathbb{R}^3$ is said to be a ***limit point*** of a set $S \subseteq \mathbb{R}^3$ if every neighbourhood of (a, b, c) contains infinitely many points of S.

The set of all limit points of S is called ***derived set*** of S and is denoted by S^d or S'.

1.9.1 Examples

1. Find the limit points and derived set of $S = \{(a, 0): a \in \mathbb{R}\}$.



→ Let $(a, 0) \in S$ be an arbitrary point. Then any nbd of $(a, 0)$ contains infinitely many points of S. Therefore $(a, 0)$ is a limit point of S. Since $(a, 0)$ is arbitrary, therefore every point of S is a limit point of S.

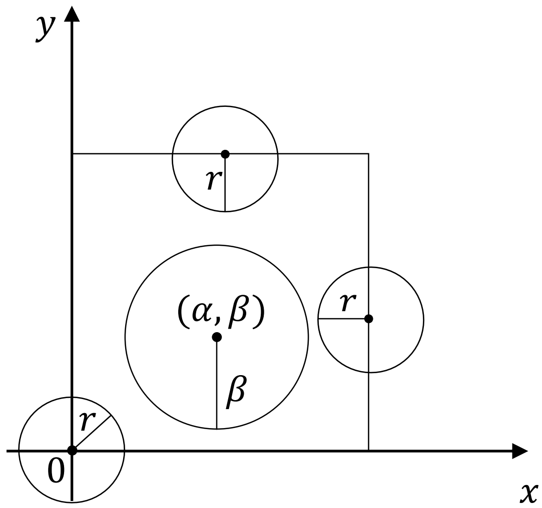
We now consider a point (α, β) where $\beta \neq 0$. Then $(\alpha, \beta) \notin S$. Now the nbd $N = \{(x, y) \in \mathbb{R}^2: (x - \alpha)^2 + (y - \beta)^2 < \beta^2\}$ does not contain any point of S. Hence (α, β) is not a limit point of S.

Therefore all limit points of S belong to S, i.e. $S' = S$.

2. Find the limit points and derived set of $S = \{(x, y) \in \mathbb{R}^2: 0 < x < 1, 0 < y < 1\}$.

→ Let $(a, b) \in S$ be an arbitrary point. Then any nbd of (a, b) contains infinitely many points of S. Therefore (a, b) is a limit point of S. Since (a, b) is arbitrary, therefore every point of S is a limit point of S.

Next we consider the origin $(0,0)$. Then the nbd $N = \{(x,y): x^2 + y^2 < r^2\}$ where r is any positive integer, contains infinitely many points of S. Thus $(0,0)$ is also a limit point of S.



Similarly if we consider any point (x,y) on the sides of the square $x = 0, x = 1, y = 0, y = 1$ and for each point, we consider a disc of radius $r > 0$ about the point (x,y), we find that each disc contains infinitely many points of S. Therefore each point on the sides of the above square is a limit pint of S.

Therefore the derived set of S is

$$S' = \{(x,y) \in \mathbb{R}^2: 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

3. Find the limit points and derived set of $S = \mathbb{Q} \times \mathbb{Q}$.

Let $(a,b) \in \mathbb{R}^2$ be an arbitrary point. Then any nbd of (a,b) contains infinitely many points of S. Therefore (a,b) is a limit point of S. Since (a,b) is arbitrary, therefore every point of $\mathbb{R}^2$ is a limit point of S.

Therefore the derived set of S is $S' = \mathbb{R}^2$.

4. Find the derived set of $S = \mathbb{Z} \times \mathbb{Z}$.

→ Let $(a,b) \in \mathbb{R}^2$ be an arbitrary point. Then there exist integers m and n such that $m \leq a < m+1$ and $n \leq b < n+1$.

Now a and b can either be the midpoints of the intervals $(m, m+1)$ and $(n, n+1)$ respectively or they are closer to any one of their respective end points. We choose the nbd

$$N = \left\{(x,y) \in \mathbb{R}^2: (x-a)^2 + (y-b)^2 < \frac{1}{4}\right\}$$

If a and b are midpoints of heir respective intervals, then N contains no point of S.

If a and b are not the midpoints, then N contains either (m,n) or $(m+1,n)$, or $(m,n+1)$ or $(m+1,n+1)$.

Thus N contains either no point of S or at most one point of S.

Hence (a,b) is not a limit point of S. Since (a,b) is arbitrary, therefore S has no limit point and consequently, the derived set of S is $S' = \phi$.

5. Find the derived set of $S = T \times T$ where T is a finite set.

→ Let $(a,b) \in S$ be an arbitrary point. Since T is a finite set, so is $T \times T$. Thus any nbd of (a,b) contains finitely many points of S. Hence (a,b) is not a limit point of S. Since (a,b) is arbitrary, S has no limit point and consequently, the derived set of S is $S' = \phi$.

6. Find the derived set of $S = \left\{\left(\frac{1}{m}, \frac{1}{n}\right) \in \mathbb{R}^2 : m, n \in \mathbb{N}\right\}$.

→ Let $m \in \mathbb{N}$ and $\epsilon > 0$ be arbitrary.

By Archimedean property of $\mathbb{R}$, there exists a natural number p such that

$$0 < \frac{1}{p} < \epsilon \quad \text{and} \quad 0 < \cdots < \frac{1}{p+2} < \frac{1}{p+1} < \frac{1}{p} < \epsilon$$

$$\therefore \left(\frac{1}{m} - \frac{1}{m}\right)^2 + \left(\frac{1}{k} - 0\right)^2 = \frac{1}{k^2} < \epsilon^2 \quad \text{for} \quad k = p, p+1, p+2, \ldots$$

Therefore an arbitrary nbd N of $\left(\frac{1}{m}, 0\right)$, defined by

$$N = \left\{(x,y) \in \mathbb{R}^2 : \left(x - \frac{1}{m}\right)^2 + (y-0)^2 < \epsilon^2\right\}$$

contains infinitely many points of S. Hence $\left(\frac{1}{m}, 0\right)$ is a limit point of S.

Similarly, we can show that $\left(0, \frac{1}{n}\right)$ is a limit point of S.

Again by Archimedean property of $\mathbb{R}$, there exists a natural number p such that

$$0 < \frac{1}{p} < \frac{\epsilon}{\sqrt{2}} \quad \text{and} \quad 0 < \cdots < \frac{1}{p+2} < \frac{1}{p+1} < \frac{1}{p} < \frac{\epsilon}{\sqrt{2}}$$

$$\therefore \left(\frac{1}{k} - 0\right)^2 + \left(\frac{1}{k} - 0\right)^2 = \frac{2}{k^2} < \epsilon^2 \quad \text{for} \quad k = p, p+1, p+2, \ldots$$

This shows that the nbd N of $(0,0)$ contains infinitely many points of S. Hence $(0,0)$ is a limit point of S.

Thus the derived set of S is

$$S' = \{(0,0)\} \cup \left\{\left(\frac{1}{m}, 0\right) : m \in \mathbb{N}\right\} \cup \left\{\left(0, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$$

➢ **Theorem 1.9.1: Let $F \subseteq \mathbb{R}^3$. F is closed in $\mathbb{R}^3$ iff $F' \subseteq F$.**

Proof: Let F be a closed set in $\mathbb{R}^3$ and $X \in \mathbb{R}^3 - F$. Then $X \in F^c$ in $\mathbb{R}^3$. As F^c is open, X is an interior point of F^c in $\mathbb{R}^3$. Hence there exists a nbd N of X such that $N \subseteq F^c$, i.e. $N \cap F = \phi$.

This shows that X is not a limit point of F. Hence all limit points of F belong to F in $\mathbb{R}^3$, i.e. $F' \subseteq F$.

Conversely, let $F' \subseteq F$ and $X \in F^c$ in $\mathbb{R}^3$.

Clearly X is not a limit point of F, as $F' \subseteq F$.

Hence there exists a nbd N of X such that N contains at most a finite number of points $X_1, X_2, \dots, X_k$ of F.

Let $X = (a, b, c)$ and $X_i = (a_i, b_i, c_i)$ for $i = 1,2, \dots, k$. Let

$$r_i = \sqrt{(a - a_i)^2 + (b - b_i)^2 + (c - c_i)^2} \qquad \text{for} \quad i = 1,2, \dots, k$$

i.e. r_i is the distance between X and X_i for $i = 1,2, \dots, k$.

Let $r = \min\{r_1, r_2, \dots, r_k\}$.

Then the nbd $\bar{N} = \{(x, y, z) \in \mathbb{R}^3 : (x - a)^2 + (y - b)^2 + (z - c)^2 < r^2\}$ does not contain $X_1, X_2, \dots, X_k$.

Hence $\bar{N}$ does not contain any point of F.

Therefore $\bar{N} \subseteq F^c$ in $\mathbb{R}^3$.

This shows that X is an interior point of F^c.

Hence F^c is an open set and so F is a closed set in $\mathbb{R}^3$.

1.10 Bolzano-Weierstrass Theorem

Every bounded infinite subset of $\mathbb{R}^3$ has a limit point in $\mathbb{R}^3$.

- **Note:** The following sets in $\mathbb{R}^2$ are closed in $\mathbb{R}^2$ as $F' \subseteq F$.

(i) $F = \{(a, 0) : a \in \mathbb{R}\}$

(ii) $F = \mathbb{Z} \times \mathbb{Z}$

(iii) $F = T \times T$ where T is a finite set.

1.11 Exercise

1. Show that the following sets are neither closed nor open sets in $\mathbb{R}^2$.

(i) $\boldsymbol{S = \{(x, y) \in \mathbb{R}^2 : 0 \le x < 1, 0 \le y < 1\}}$

(ii) $\boldsymbol{S = \{(x, y) \in \mathbb{R}^2 : y \ge 0\}}$

(iii) $\boldsymbol{S = \mathbb{Q} \times \mathbb{Q}}$

(iv) $\boldsymbol{S = \{(x, y) \in \mathbb{R}^2 : y \ge x^2, |x| < 2\}}$

1.2. Limit and continuity in $\mathbb{R}^n$

Let $D \subseteq \mathbb{R}^n$ be a non-empty set and $f: D \to \mathbb{R}$ be a function on D, i.e. $(x_1, x_2, \ldots, x_n) \in D$ implies $f(x_1, x_2, \ldots, x_n) \in \mathbb{R}$. Then f is said to be a ***real valued function*** of n variables on D. If $D \subseteq \mathbb{R}^2$, f is called a function of two variables and if $D \subseteq \mathbb{R}^3$, f is called a function of three variables.

2.1 Limit of a function

2.1.1 Limit of a function in $\mathbb{R}^2$

Let $D \subseteq \mathbb{R}^2$, $f: D \to \mathbb{R}$ be a function and (a, b) be a limit point of D. A real number l is said to be a ***limit*** or ***double limit*** of f at (a, b) if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x, y) - l| < \epsilon \quad \text{for all} \quad (x, y) \in N'_\delta(a, b) \cap D$$

where $N'_\delta(a, b)$ denotes the deleted δ-neighbourhood of (a, b). We write it as

$$\lim_{(x,y) \to (a,b)} f(x, y) = l$$

2.1.2 Limit of a function in $\mathbb{R}^3$

Let $D \subseteq \mathbb{R}^3$, $f: D \to \mathbb{R}$ be a function and (a, b, c) be a limit point of D. A real number l is said to be a ***limit*** or ***double limit*** of f at (a, b, c) if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x, y, z) - l| < \epsilon \quad \text{for all} \quad (x, y, z) \in N'_\delta(a, b, c) \cap D$$

where $N'_\delta(a, b, c)$ denotes the deleted δ-neighbourhood of (a, b, c). We write it as $\lim_{(x,y,z) \to (a,b,c)} f(x, y, z) = l$

2.1.3 Repeated limit

Let $D \subseteq \mathbb{R}^2$, $f: D \to \mathbb{R}$ be a function and (a, b) be a limit point of D. If $\lim\limits_{x \to a} f(x, y)$ exists for all y, then it is a function of y, say $g(y)$. If $\lim\limits_{y \to b} g(y)$ exists, then $\lim\limits_{y \to b} \lim\limits_{x \to a} f(x, y)$ exists and it is called a ***repeated limit*** of f.

Again if $\lim\limits_{y \to b} f(x, y)$ exists for all x, then it is a function of$x$, say $h(x)$. If $\lim\limits_{x \to a} h(x)$ exists, then $\lim\limits_{x \to a} \lim\limits_{y \to b} f(x, y)$ exists and it is called a ***repeated limit*** of f.

2.1.3.1 Worked example

1. Let

$$\boldsymbol{f(x, y) = \frac{x \sin\left(\frac{1}{x}\right) + y}{x + y}, \qquad x + y \neq 0}$$

Show that $\boldsymbol{\lim\limits_{y \to 0} \lim\limits_{x \to 0} f(x, y) = 1}$, but $\boldsymbol{\lim\limits_{x \to 0} \lim\limits_{y \to 0} f(x, y)}$ does not exists.

→ We have

$$f(x, y) = \frac{x \sin\left(\frac{1}{x}\right) + y}{x + y}, \qquad x + y \neq 0$$

Now

$$\lim_{x \to 0} f(x, y)$$

$$= \lim_{x \to 0} \frac{x \sin\left(\frac{1}{x}\right) + y}{x + y} \quad \text{[for fixed } y\text{]}$$

$$= \frac{y}{y} \left[\because \sin\left(\frac{1}{x}\right) \text{ is a bounded function on } N'(0) \text{ and } \lim_{x \to 0} x = 0, \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0 \right]$$

$$= 1$$

Therefore

$$\lim_{y \to 0} \lim_{x \to 0} f(x, y) = \lim_{y \to 0} 1 = 1$$

Again $\lim\limits_{y \to 0} f(x, y) = \lim\limits_{y \to 0} \frac{x \sin\left(\frac{1}{x}\right) + y}{x + y}$ [for fixed x] $= \sin\left(\frac{1}{x}\right)$

Since $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist, by Cauchy's condition, therefore $\lim_{x\to 0}\lim_{y\to 0} \sin\left(\frac{1}{x}\right)$ does not exit.

2. Let

$$\boldsymbol{f(x,y) = \frac{x - y + x^2 + y^2}{x + y}, \qquad x > 0, \qquad y > 0}$$

Show that both repeated limit exist but they are different.

3. Let $\boldsymbol{f(x,y) = x \sin\left(\frac{1}{y}\right),\ y \neq 0}$. Show that $\boldsymbol{\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 1}$ but $\boldsymbol{\lim_{x\to 0}\lim_{y\to 0} f(x,y)}$ does not exists.

4. Let

$$\boldsymbol{f(x,y) = \frac{\sin x + \sin 2y}{\tan 2x + \tan y}, \qquad (x,y) \neq (0,0)}$$

Show that $\boldsymbol{\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 2}$ and $\boldsymbol{\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \frac{1}{2}}$.

5. Let

$$\boldsymbol{f(x,y) = xy\frac{x^2 - y^2}{x^2 + y^2}, \qquad (x,y) \neq (0,0)}$$

Show that $\boldsymbol{\lim_{(x,y)\to(0,0)} f(x,y) = 0}$.

→ Let $\epsilon > 0$ be arbitrary.

We note that

$$\left|\frac{x^2 - y^2}{x^2 + y^2}\right| \leq 1, \qquad |x| \leq \sqrt{x^2 + y^2}, \qquad |y| \leq \sqrt{x^2 + y^2}$$

Now

$$\begin{aligned} |f(x,y) - f(0,0)| &= \left|xy\frac{x^2 - y^2}{x^2 + y^2}\right| \\ &\leq |xy| \\ &= |x||y| \\ &\leq x^2 + y^2 < \epsilon \end{aligned}$$

holds if $(x - 0)^2 + (y - 0)^2 < \delta^2 = \epsilon$ where $\delta = \sqrt{\epsilon}$.

Thus for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x,y) - f(0,0)| < \epsilon$ when $(x-0)^2 + (y-0)^2 < \delta^2$.

This shows that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

6. Let

$$\boldsymbol{f(x,y) = \frac{x^3 - y^3}{x^2 + y^2}, \qquad (x,y) \neq (0,0)}$$

Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

→ Let $\epsilon > 0$ be arbitrary.

We note that

$$|x^3 - y^3| \leq |x^3 + y^3|, \qquad |x^3| \leq (x^2 + y^2)^{\frac{3}{2}}, \qquad |y^3| \leq (x^2 + y^2)^{\frac{3}{2}}$$

Now

$$\begin{aligned} |f(x,y) - f(0,0)| &= \left|\frac{x^3 - y^3}{x^2 + y^2}\right| \\ &\leq \left|\frac{x^3 + y^3}{x^2 + y^2}\right| \\ &\leq \frac{2(x^2 + y^2)^{\frac{3}{2}}}{x^2 + y^2} \\ &= 2\sqrt{x^2 + y^2} < \epsilon \end{aligned}$$

holds if $\sqrt{(x-0)^2 + (y-0)^2} < \delta = \epsilon/2$ where $\delta = \epsilon/2$.

Thus for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x,y) - f(0,0)| < \epsilon$ when $\sqrt{(x-0)^2 + (y-0)^2} < \delta$.

This shows that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

7. Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ where

(i) $\boldsymbol{f(x,y) = \frac{x^2y^2}{x^2+y^2},\ (x,y) \neq (0,0)}$

(ii) $\boldsymbol{f(x,y) = \frac{x^3y^3}{x^2+y^2},\ (x,y) \neq (0,0)}$

(iii) $\boldsymbol{f(x,y) = \frac{x^4+y^4}{x^2+y^2},\ (x,y) \neq (0,0)}$

→ (iii) Let $\epsilon > 0$ be arbitrary.

Now

$$|f(x,y) - f(0,0)| = \left|\frac{x^4 + y^4}{x^2 + y^2}\right|$$

$$\leq \left|\frac{x^4}{x^2 + y^2}\right| + \left|\frac{y^4}{x^2 + y^2}\right|$$

$$\leq \left|\frac{(x^2 + y^2)^2}{x^2 + y^2}\right| + \left|\frac{(x^2 + y^2)^2}{x^2 + y^2}\right|$$

$$= 2(x^2 + y^2) < \epsilon$$

holds if $(x - 0)^2 + (y - 0)^2 < \delta^2 = \epsilon/2$ where $\delta = \sqrt{\epsilon/2}$

Thus for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x,y) - f(0,0)| < \epsilon$ when $(x - 0)^2 + (y - 0)^2 < \delta^2$.

This shows that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

8. Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist where

(i) $f(x,y) = \frac{x^2-y^2}{x^2+y^2},\ (x,y) \neq (0,0)$

(ii) $f(x,y) = \frac{xy^2}{x^2+y^4},\ (x,y) \neq (0,0)$

(iii) $f(x,y) = \frac{x^3+y^3}{x-y},\ (x,y) \neq (0,0)$

(iv) $f(x,y) = \frac{x^2y}{x^4+y^2},\ (x,y) \neq (0,0)$

→

(i) Let $(x,y) \to (0,0)$ along the path $y = mx$. Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2}$$

$$= \frac{1 - m^2}{1 + m^2} \quad [\because x \to 0, x \neq 0]$$

which is different for different values of m. Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(ii) Choose the path $x = my^2$.

(iii) First we let $(x,y) \to (0,0)$ along the path $y = 0$. Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3}{x} = 0$$

Now we let $(x,y) \to (0,0)$ along the path $y = \sin x$. then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3 + \sin^3 x}{x - \sin x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{(x,y)\to(0,0)} \frac{3x^2 + 3\sin^2 x \cos x}{1 - \cos x} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{(x,y)\to(0,0)} \frac{6x - 3\sin^3 x + 6\sin x \cos^2 x}{\sin x}$$

$$= \lim_{(x,y)\to(0,0)} \left[6\left(\frac{x}{\sin x}\right) - 3\sin^2 x + 6\cos^2 x\right]$$

$$= 6 + 6 \qquad \left[\because \lim_{x\to 0}\left(\frac{x}{\sin x}\right) = 1\right]$$

$$= 12$$

Thus we see that the limiting values are different for two different paths to the origin. Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Alternative method:

(iv) let $(x,y) \to (0,0)$ along the path $x - y = mx^3$. Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^3 + (x + mx^3)^3}{mx^3}$$

$$= \lim_{x\to 0} \frac{1 + (1 + mx^2)^3}{m}$$

$$= \frac{2}{m}$$

which is different for different values of m. Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(v) Choose the path $y = mx^2$.

- **Note:** If two repeated limits exist and not equal, then the double limit does not exist.

9. Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist where

(i) $f(x,y) = \frac{\sin x+\sin 2y}{\tan 2x+\tan y},\ (x,y) \neq (0,0)$

(ii) $f(x,y) = \frac{x-y+x^2+y^2}{x+y},\ (x,y) \neq (0,0)$

[Hint: Here two repeated limits are unequal.]

10. Let

$$f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x}, & xy \neq 0 \\ 0 \qquad\qquad\qquad\quad , & xy = 0 \end{cases}$$

Show that the double limit of f at $(0,0)$ exists but both the repeated limits do not exist at $(0,0)$.

→ Let $\epsilon > 0$ be arbitrary.

Now

$$\begin{aligned} |f(x,y) - f(0,0)| &= \left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right| \\ &\leq \left|x\sin\frac{1}{y}\right| + \left|y\sin\frac{1}{x}\right| \\ &\leq |x| + |y| \\ &\leq 2\sqrt{x^2+y^2} < \epsilon \end{aligned}$$

holds if $(x-0)^2 + (y-0)^2 < \delta^2 = \epsilon^2/4$ where $\delta = \epsilon/2$.

Thus for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x,y) - f(0,0)| < \epsilon$ when $(x-0)^2 + (y-0)^2 < \delta^2$.

This shows that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

Since $\lim\limits_{y\to0} \sin\frac{1}{y}$ does not exist, therefore $\lim\limits_{y\to0} f(x,y)$ does not exists. Consequently, $\lim\limits_{x\to0}\lim\limits_{y\to0} f(x,y)$ does not exists.

Again $\lim\limits_{x\to0} \sin\frac{1}{x}$ does not exist, therefore $\lim\limits_{x\to0} f(x,y)$ does not exists. Consequently, $\lim\limits_{y\to0}\lim\limits_{x\to0} f(x,y)$ does not exists.

11. Let

$$\boldsymbol{f(x,y) = \begin{cases} 1, & xy \neq 0 \\ 0, & xy = 0 \end{cases}}$$

Show that the repeated limits of $\boldsymbol{f}$ exists at $(\mathbf{0},\mathbf{0})$ and are equal but the double limit does not exist at $(\mathbf{0},\mathbf{0})$.

→ Given

$$\lim_{y\to 0} f(x,y) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Therefore $\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 1$, because $x \to 0$ implies $x \neq 0$.

Again

$$\lim_{x\to 0} f(x,y) = \begin{cases} 1, & y \neq 0 \\ 0, & y = 0 \end{cases}$$

Therefore $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 1$, because $yx \to 0$ implies $y \neq 0$.

Hence the repeated limits exist and are equal to 1.

Let $(x,y) \to (0,0)$ along the path $y = 0$, i.e. along the x-axis. Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} 0 = 0$$

Again let $(x,y) \to (0,0)$ along the path $y = x$. Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,x) = \lim_{x\to 0} 1 = 1$$

For two different paths, we get two different limits. Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exit.

- **Note: The above example shows that even if both the repeated limits exist and are equal, the double limit may not exist.**

12. Let

$$\boldsymbol{f(x,y) = \frac{xy}{x^2+y^2}, \qquad (x,y) \neq (0,0)}$$

Show that the repeated limits of $\boldsymbol{f}$ exists at $(\mathbf{0},\mathbf{0})$ and are equal but the double limit does not exist at $(\mathbf{0},\mathbf{0})$.

13. Let

$$\boldsymbol{f(x,y)=\begin{cases} x\sin\frac{1}{y}+\frac{x^2-y^2}{x^2+y^2}, & y\neq 0 \\ 0, & y=0 \end{cases}}$$

Show that $\boldsymbol{\lim_{y\to 0}\lim_{x\to 0} f(x,y)}$ exists but neither $\boldsymbol{\lim_{x\to 0}\lim_{y\to 0} f(x,y)}$ nor $\boldsymbol{\lim_{(x,y)\to(0,0)} f(x,y)}$ exist.

→

$$\lim_{x\to 0} f(x,y)=\lim_{x\to 0}\left(x\sin\frac{1}{y}+\frac{x^2-y^2}{x^2+y^2}\right)$$

$$=-\frac{y^2}{y^2}\quad\left[\because\ \lim_{x\to 0}\left(x\sin\frac{1}{y}\right)=0\right]$$

$$=-1$$

$$\therefore\ \lim_{y\to 0}\lim_{x\to 0} f(x,y)=\lim_{y\to 0}(-1)=-1$$

Hence $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ exists.

Since $\lim_{y\to 0}(x\sin\frac{1}{y})$ does not exist, by Cauchy's condition, therefore $\lim_{y\to 0} f(x,y)$ does not exist and consequently $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ does not exist.

Let $f(x,y)=g(x,y)+h(x,y)$

where

$$g(x,y)=\begin{cases} x\sin\frac{1}{y}, & y\neq 0 \\ 0, & y=0 \end{cases}$$

and

$$h(x,y)=\begin{cases} \frac{x^2-y^2}{x^2+y^2}, & xy\neq 0 \\ 0, & xy=0 \end{cases}$$

Now

$$\lim_{(x,y)\to(0,0)} g(x,y)=\lim_{(x,y)\to(0,0)} x\sin\frac{1}{y}=0$$

Let $(x, y) \to (0,0)$ along the path $y = mx$. Then

$$\lim_{(x,y)\to(0,0)} h(x,y) = \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1 - m^2}{1 + m^2}$$

which is different for different values of m. Hence $\lim_{(x,y)\to(0,0)} h(x,y)$ does not exist and consequently, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

14. Let

$$\boldsymbol{f(x,y) = \begin{cases} y \sin\frac{1}{x} + \frac{xy}{x^2 + y^2}, & x \neq 0 \\ 0 \quad , & x = 0 \end{cases}}$$

Show that $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ exists but neither $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ nor $\lim_{(x,y)\to(0,0)} f(x,y)$ exist.

➢ **Theorem 1.3.1:** Let $D \subseteq \mathbb{R}^2$, $f: D \to \mathbb{R}$ be a function and (a, b) be a limit point of D. Let $\lim_{(x,y)\to(a,b)} f(x,y)$ exist and equal to A. Let $\lim_{x\to a} f(x,y)$ exist for each fixed value of y in the deleted neighbourhood of b and $\lim_{y\to b} f(x,y)$ exist for each fixed value of x in the deleted neighbourhood of a. Then $\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y) = A$.

Proof: Since $\lim_{x\to a} f(x,y)$ exists, let $\lim_{x\to a} f(x,y) = g(y)$.

Let $\epsilon > 0$ be arbitrary. Then for fixed value of y, there exists $\delta_1 > 0$ such that

$$|f(x,y) - g(y)| < \frac{\epsilon}{2} \quad \text{for all } x \text{ satisfying } 0 < |x - a| < \delta_1 \qquad (1)$$

Again $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and equal to A. Then for the above chosen $\epsilon > 0$, there exists $\delta_2 > 0$ such that

$$|f(x,y) - A| < \frac{\epsilon}{2} \ \forall\, x, y \text{ satisfying } 0 < |x - a| < \delta_1, 0 < |y - b| < \delta_2 \qquad (2)$$

Now (1) holds for each fixed value of y. Therefore for $0 < |y-b| < \delta_2$, there exists $\delta_3 > 0$ such that

$$|f(x,y) - g(y)| < \frac{\epsilon}{2} \quad \text{for all } x \text{ satisfying } 0 < |x-a| < \delta_3 \qquad (3)$$

Let $\delta = \min\{\delta_2, \delta_3\} > 0$. Then (2) and (3) hold for all x satisfying $0 < |x-a| < \delta, 0 < |y-b| < \delta$. Now

$$\begin{aligned} |g(y) - A| &= |f(x,y) - A + g(y) - f(x,y)| \\ &\le |f(x,y) - A| + |f(x,y) - g(y)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all y satisfying $0 < |y-b| < \delta$.

This shows that $\lim_{y\to b} g(y) = A$, i.e. $\lim_{y\to b}\lim_{x\to a} f(x,y) = A$.

Similarly we can show that $\lim_{x\to a}\lim_{y\to b} f(x,y) = A$.

Thus $\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y) = A$.

This completes the proof.

1.3.Continuity of a function

3.1 Continuity in $\mathbb{R}^2$

Let $D \subseteq \mathbb{R}^2$, $f: D \to \mathbb{R}$ be a function and $(a,b) \in D$. Then f is said to be ***continuous*** at (a,b) if for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x,y) - f(a,b)| < \epsilon$ for all $(x,y) \in N'_\delta(a,b) \cap D$

i.e., $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

i.e., $\lim_{(h,k)\to(00)} f(a+h, b+k) = f(a,b)$ where $(a+h, b+k) \in D$.

3.2 Continuity in $\mathbb{R}^3$

Let $D \subseteq \mathbb{R}^3$, $f: D \to \mathbb{R}$ be a function and $(a,b,c) \in D$. Then f is said to be ***continuous*** at (a,b,c) if for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x,y,z) - f(a,b,c)| < \epsilon$ for all $(x,y,z) \in N'_\delta(a,b,c) \cap D$

i.e., $\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c)$.

3.3 Exercise

1. Show that f is continuous at $(0,0)$ where

$$(i)\qquad f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2}, & x^2+y^2\neq 0\\ 0\quad , & x^2+y^2=0\end{cases}$$

$$(ii)\qquad f(x,y)=\begin{cases}(px+qy)\sin\dfrac{x}{y}, & y\neq 0\\ 0\quad , & y=0\end{cases}$$

$$(iii)\qquad f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0\quad , & (x,y)=(0,0)\end{cases}$$

$$(iv)\qquad f(x,y)=\begin{cases}xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0\quad , & (x,y)=(0,0)\end{cases}$$

→

(i) Let $\epsilon>0$ be arbitrary. Then

$$|f(x,y)-f(0,0)|$$

$$=\left|\frac{x^3-y^3}{x^2+y^2}\right|$$

$$\leq\frac{|x^3|}{x^2+y^2}+\frac{|y^3|}{x^2+y^2}$$

$$=\frac{|x|^3}{x^2+y^2}+\frac{|y|^3}{x^2+y^2}$$

$$\leq\frac{(x^2+y^2)^{\frac{3}{2}}}{x^2+y^2}+\frac{(x^2+y^2)^{\frac{3}{2}}}{x^2+y^2}\quad\left[\because\ |x|\leq\sqrt{x^2+y^2},|y|\leq\sqrt{x^2+y^2}\right]$$

$$\leq\frac{2(x^2+y^2)^{\frac{3}{2}}}{x^2+y^2}$$

$$=2\sqrt{x^2+y^2}<\epsilon$$

holds if $(x-0)^2+(y-0)^2<\delta^2=\epsilon^2/4$ where $\delta=\epsilon/2$.

Therefore $|f(x,y)-f(0,0)|<\epsilon$ whenever $(x-0)^2+(y-0)^2<\delta^2$ holds.

This shows that $f(x,y)$ is continuous at $(0,0)$.

(ii) Let $\epsilon>0$ be arbitrary. Then

$$\begin{aligned}
&|f(x,y)-f(0,0)| \\
&= \left|(px+qy)\sin\frac{x}{y}\right| \\
&\le |(px+qy)| \quad [\because \left|\sin\frac{x}{y}\right| \le 1] \\
&\le |px|+|qy| \\
&= |p||x|+|q||y| \\
&\le (|p|+|q|)\sqrt{x^2+y^2}<\epsilon
\end{aligned}$$

holds if $(x-0)^2+(y-0)^2<\delta^2=\frac{\epsilon^2}{(|p|+|q|)^2}$ where $\delta=\epsilon/(|p|+|q|)$.

Therefore $|f(x,y)-f(0,0)|<\epsilon$ whenever $(x-0)^2+(y-0)^2<\delta^2$ holds.

This shows that $f(x,y)$ is continuous at $(0,0)$.

(iii) Hint: $|x|\le\sqrt{x^2+y^2}$ and $|y|\le\sqrt{x^2+y^2}$. So $\frac{xy}{\sqrt{x^2+y^2}}\le\sqrt{x^2+y^2}$.

(iv) Hint: $|x|\le\sqrt{x^2+y^2}$, $|y|\le\sqrt{x^2+y^2}$ and $\left|\frac{x^2-y^2}{x^2+y^2}\right|\le 1$. So $\left|xy\frac{x^2-y^2}{x^2+y^2}\right|\le |xy|\le x^2+y^2$.

2. Show that f is not continuous at $(0,0)$ where

$$(i) \qquad f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^4+y^4}}, & (x,y)\ne(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$$(ii) \qquad f(x,y)=\begin{cases}\dfrac{x^4+y^4}{x-y}, & x\ne y\\ 0, & x=y\end{cases}$$

$$(iii) \qquad f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x-y}, & x\ne y\\ 0, & x=y\end{cases}$$

$$(iv) \qquad f(x,y)=\begin{cases}\dfrac{\sin(x^4+y^4)}{(x^4+y^4)}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

→

(i) Let $(x,y)\to(0,0)$ along the path $y=mx$. Then

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{x\to 0}\frac{mx^2}{\sqrt{x^4+m^4x^4}}=\frac{m}{\sqrt{1+m^4}}$$

which is different for different values of m. Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Therefore f is not continuous at $(0,0)$.

(ii) Let $(x,y)\to(0,0)$ along the path $x-y=mx^4$. Then

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{x\to 0}\frac{x^4+(x-mx^4)^4}{mx^4}=\lim_{x\to 0}\frac{1+(1-mx^3)^4}{m}=\frac{2}{m}$$

which is different for different values of m. Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Therefore f is not continuous at $(0,0)$.

(iii) Hint: Choose the path $x-y=mx^3$.

(iv) Hint: Choose the path $y=mx$ and applying L'Hospital rule, obtain the limit to be 1. Show $\lim_{(x,y)\to(0,0)} f(x,y)\neq f(0,0)$.

3. Show that f is not continuous at $(0,0)$ where

$$(i) \qquad f(x,y)=\begin{cases}\dfrac{xy+z}{x^2+y^2+z^2}, & (x,y,z)\neq(0,0)\\ 0, & (x,y,z)=(0,0)\end{cases}$$

$$(ii) \qquad f(x,y)=\begin{cases}\dfrac{x^2-y^2+z^2}{x^2+y^2+z^2}, & (x,y,z)\neq(0,0)\\ 0, & (x,y,z)=(0,0)\end{cases}$$

→

(i) Hint: Choose the path $y=mx, z=0$.

(ii) Hint: Choose the path $y=mx, z=0$.

➢ **Theorem 3.1: Let $\boldsymbol{D} \subseteq \mathbb{R}^2$, $\boldsymbol{f}: \boldsymbol{D} \to \mathbb{R}$ be a function continuous at $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{D}$. Then there exists a neighbourhood of $(\boldsymbol{a}, \boldsymbol{b})$ in which $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$ is bounded.**

Proof: Since f is continuous at (a, b), therefore for $\epsilon = 1$, there exists $\delta > 0$ such that

$$|f(x,y) - f(a,b)| < 1 \quad \text{for all } (x,y) \in N'_\delta(a,b) \cap D$$

$$\Rightarrow f(a,b) - 1 < f(x,y) < f(a,b) + 1 \quad \text{for all } (x,y) \in N'_\delta(a,b) \cap D$$

Let $m = f(a,b) - 1$ and $M = f(a,b) + 1$. Then $m, M \in \mathbb{R}$. Thus

$$m < f(x,y) < M \quad \text{for all } (x,y) \in N'_\delta(a,b) \cap D$$

This shows that f is bounded in $N'_\delta(a,b) \cap D$.

This completes the proof.

- **Note:** Consider the function

$$f(x,y) = \begin{cases} \dfrac{1}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0 \quad , & x^2 + y^2 \neq 0 \end{cases}$$

Since f is not bounded in every neighbourhood of $(0,0)$, therefore f is not continuous at $(0,0)$.

➢ **Theorem 3.2: Let $\boldsymbol{D} \subseteq \mathbb{R}^2$, $\boldsymbol{f}, \boldsymbol{g}: \boldsymbol{D} \to \mathbb{R}$ be two functions continuous at $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{D}$. Then**

(i) $|\boldsymbol{f}|$ is continuous at $(\boldsymbol{a}, \boldsymbol{b})$.

(ii) $\boldsymbol{cf}$ is continuous at $(\boldsymbol{a}, \boldsymbol{b})$ where $\boldsymbol{c} \neq \boldsymbol{0}$.

(iii) $\boldsymbol{f} \pm \boldsymbol{g}$ is continuous at $(\boldsymbol{a}, \boldsymbol{b})$.

(iv) $\boldsymbol{fg}$ is continuous at $(\boldsymbol{a}, \boldsymbol{b})$.

(v) $\boldsymbol{f}/\boldsymbol{g}$ is continuous at $(\boldsymbol{a}, \boldsymbol{b})$ provided $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) \neq \boldsymbol{0}$ in $\boldsymbol{D}$.

Proof:

(i) $f: D \to \mathbb{R}$ is defined by $|f|(x,y) = |f(x,y)|, (x,y) \in D$.

$$\big||f|(x,y) - |f|(a,b)\big| = \big||f(x,y)| - |f(a,b)|\big| \leq |f(x,y) - f(a,b)|$$

Since f is continuous at (a, b), therefore for arbitrarily chosen $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x,y) - f(a,b)| < \epsilon \quad \text{for all} \quad (x,y) \in N'_\delta(a,b) \cap D$$

Therefore $\big||f|(x,y) - |f|(a,b)\big| \leq |f(x,y) - f(a,b)| < \epsilon$ for all $(x,y) \in N'_\delta(a,b) \cap D$.

This shows that $|f|$ is continuous at (a,b).

(iv) We have

$$|fg(x,y) - fg(a,b)|$$
$$= |f(x,y)g(x,y) - f(a,b)g(a,b)|$$
$$= |f(x,y)\{g(x,y) - g(a,b)\} - g(a,b)\{f(x,y) - f(a,b)\}|$$
$$\leq |f(x,y)||g(x,y) - g(a,b)| + |g(a,b)||f(x,y) - f(a,b)| \qquad (1)$$

Let $\epsilon > 0$ be arbitrary.

Since $f(x,y)$ is continuous at (a,b), therefore for the above chosen $\epsilon > 0$, there exists $\delta_1 > 0$ such that

$$|f(x,y) - f(a,b)| < \epsilon \quad \text{for all} \quad (x,y) \in N'_{\delta_1}(a,b) \cap D \qquad (2)$$

Since $\big||f(x,y)| - |f(a,b)|\big| \leq |f(x,y) - f(a,b)|$, it follows that

$$\big||f(x,y)| - |f(a,b)|\big| < \epsilon \quad \text{for all} \quad (x,y) \in N'_{\delta_1}(a,b) \cap D$$

$$\Rightarrow |f(a,b)| - \epsilon < |f(x,y)| < |f(a,b)| + \epsilon \quad \text{for all} \quad (x,y) \in N'_{\delta_1}(a,b) \cap D$$

Let $B' = |f(a,b)| + \epsilon$. Then $|f(x,y)| < B'$ for all $(x,y) \in N'_{\delta_1}(a,b) \cap D$.

Let $B = \max\{B', |g(a,b)|\} > 0$. Therefore, from (1)

$$|fg(x,y) - fg(a,b)|$$
$$< B|g(x,y) - g(a,b)| + B|f(x,y) - f(a,b)| \qquad (3)$$

Since $f(x,y)$ is continuous at (a,b), therefore for the above chosen $\epsilon > 0$, there exists $\delta_2 > 0$ such that

$$|f(x,y) - f(a,b)| < \frac{\epsilon}{2B} \quad \text{for all} \quad (x,y) \in N'_{\delta_2}(a,b) \cap D \qquad (4)$$

Since $g(x,y)$ is continuous at (a,b), therefore for the above chosen $\epsilon > 0$, there exists $\delta_3 > 0$ such that

$$|g(x,y)-g(a,b)|<\frac{\epsilon}{2B} \quad \text{for all } (x,y)\in N'_{\delta_3}(a,b)\cap D \tag{5}$$

Let $\delta=\min\{\delta_1,\delta_2,\delta_3\}$. Then (4) and (5) hold for all $(x,y)\in N'_\delta(a,b)\cap D$.

Thus from (3),

$$|fg(x,y)-fg(a,b)|<B.\frac{\epsilon}{2B}+B.\frac{\epsilon}{2B}=\epsilon \text{ for all } (x,y)\in N'_\delta(a,b)\cap D$$

This shows that fg is continuous at (a,b).

(v) First we prove that $1/g$ is continuos at (a,b) if $g(x,y)\neq 0$ for all $(x,y)\in D$.

$$\left|\frac{1}{g}(x,y)-\frac{1}{g}(a,b)\right|=\left|\frac{1}{g(x,y)}-\frac{1}{g(a,b)}\right|=\frac{|g(x,y)-g(a,b)|}{|g(x,y)||g(a,b)|} \tag{1}$$

Let $\epsilon=|g(a,b)|/2$. Since g is continuous at (a,b), there exists $\delta_1>0$ such that

$$|g(x,y)-g(a,b)|<\frac{|g(a,b)|}{2} \quad \text{for all } (x,y)\in N'_{\delta_1}(a,b)\cap D$$

Now

$$\big||g(x,y)|-|g(a,b)|\big|\le|g(x,y)-g(a,b)|<\frac{|g(a,b)|}{2}$$

$$\Rightarrow |g(a,b)|-\frac{|g(a,b)|}{2}<|g(x,y)|<|g(a,b)|-\frac{|g(a,b)|}{2}$$

$$\Rightarrow |g(x,y)|>\frac{|g(a,b)|}{2}$$

$$\Rightarrow |g(x,y)||g(a,b)|>\frac{|g(a,b)|^2}{2} \tag{2}$$

for all $(x,y)\in N'_{\delta_1}(a,b)\cap D$.

Let $\epsilon>0$ be arbitrary. Since g is continuous at (a,b), there exists $\delta_2>0$ such that

$$|g(x,y)-g(a,b)|<\frac{|g(a,b)|^2\epsilon}{2} \text{ for all } (x,y)\in N'_{\delta_2}(a,b)\cap D \tag{3}$$

Let $\delta=\min\{\delta_1,\delta_2\}>0$. Then, using (1), (2) and (3), we obtain

$$\left|\frac{1}{g}(x,y)-\frac{1}{g}(a,b)\right|<\epsilon \quad \text{for all } (x,y)\in N'_\delta(a,b)\cap D$$

This shows that $1/g$ is continuous at (a, b).

Since f is also continuous at (a, b), therefore f/g is also continuous at (a, b).

➢ **Theorem 3.3: Let $\boldsymbol{D} \subseteq \mathbb{R}^2$, $\boldsymbol{f}: \boldsymbol{D} \to \mathbb{R}$ be a continuous function at $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{D}$. Then the functions $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{b})$ and $\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{y})$ are continuous at $\boldsymbol{x} = \boldsymbol{a}$ and $\boldsymbol{y} = \boldsymbol{b}$ respectively.**

Proof: Since f is continuous at (a, b), therefore for arbitrarily chosen $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x, y) - f(a, b)| < \epsilon$ for $(x - a)^2 + (y - b)^2 < \delta^2$ (1)

The above relation holds for $y = b$ also. Therefore

$$|f(x, b) - f(a, b)| < \epsilon \qquad \text{for } (x - a)^2 < \delta^2$$

$$\text{i.e.} \quad |f(x, b) - f(a, b)| < \epsilon \qquad \text{for } |x - a| < \delta$$

This shows that $f(x, b)$ is continuous at $x = a$.

Similarly (1) holds for $x = a$ also. Proceeding as before, we get

$$|f(a, y) - f(a, b)| < \epsilon \qquad \text{for } |y - b| < \delta$$

This shows that $f(a, y)$ is continuous at $y = b$.

- **Note:** Converse of this theorem is not true. Consider the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0 \quad , & (x, y) = (0,0) \end{cases}$$

Here $f(x, 0) = 0 = f(0,0)$ and $f(0, y) = 0 = f(0,0)$

$\therefore$ Both $f(x, 0)$ and $f(0, y)$ are continuous at $x = 0$ and $y = 0$ respectively.

Let $(x, y) \to 0$ along the path $y = mx$. Then

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{x\to 0} \frac{mx^2}{(1 + m^2)x^2} = \frac{m}{1 + m^2}$$

which is different for different values of m. Therefore $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist. Hence f is not continuous at $(0,0)$.

➢ **Theorem 3.4: Let $\boldsymbol{D} \subseteq \mathbb{R}^2$, $\boldsymbol{f}: \boldsymbol{D} \to \mathbb{R}$ be a continuous function at $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{D}$. If $\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}) \neq 0$, then there exists a neighbourhood of $(\boldsymbol{a}, \boldsymbol{b})$ in**

which $f(x,y)$ and $f(a,b)$ have the same sign.

Proof: Since f is continuous at (a,b), therefore for arbitrarily chosen $\epsilon = |f(a,b)|/2$, there exists $\delta > 0$ such that

$$|f(x,y) - f(a,b)| < \frac{|f(a,b)|}{2} \qquad \text{for all} \quad (x,y) \in N_\delta(a,b) \cap D$$

$$\Rightarrow f(a,b) - \frac{|f(a,b)|}{2} < f(x,y) < f(a,b) + \frac{|f(a,b)|}{2}$$

$$\text{for all } (x,y) \in N_\delta(a,b) \cap D \qquad (1)$$

Two cases arise:

Case-1: $|f(a,b)| > 0$

Then (1) becomes

$$0 < \frac{f(a,b)}{2} < f(x,y) < \frac{3f(a,b)}{2} \qquad \text{for all } (x,y) \in N_\delta(a,b) \cap D$$

$$\Rightarrow f(x,y) > 0 \qquad \text{for all } (x,y) \in N_\delta(a,b) \cap D$$

Thus $f(x,y)$ and $f(a,b)$ are of same sign.

Case-2: $|f(a,b)| < 0$

Then (1) becomes

$$\frac{3f(a,b)}{2} < f(x,y) < \frac{f(a,b)}{2} < 0 \qquad \text{for all } (x,y) \in N_\delta(a,b) \cap D$$

$$\Rightarrow f(x,y) < 0 \qquad \text{for all } (x,y) \in N_\delta(a,b) \cap D$$

Thus $f(x,y)$ and $f(a,b)$ are of same sign.

This completes the proof.

1.4. Partial Derivatives

4.1 First order partial derivatives

4.1.1 In $\mathbb{R}^2$

Let $D \subseteq \mathbb{R}^2$. Let $f: D \to \mathbb{R}$ be a function and (x,y) be an interior point of D.

i) If $\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$ exists finitely, then it is called the ***first order partial derivative*** with respect to x and is denoted by $\left.\frac{\partial f}{\partial x}\right|_{(x,y)}$ or $f_x(x,y)$.

ii) If $\lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k}$ exists finitely, then it is called the ***first order partial derivative*** with respect to y and is denoted by $\left.\frac{\partial f}{\partial y}\right|_{(x,y)}$ or $f_y(x,y)$.

4.1.2 In $\mathbb{R}^3$

Let $D \subseteq \mathbb{R}^3$. Let $f: D \to \mathbb{R}$ be a function and (x,y,z) be an interior point of D.

i) If $\lim_{h\to 0}\frac{f(x+h,y,z)-f(x,y,z)}{h}$ exists finitely, then it is called the ***first order partial derivative*** with respect to x and is denoted by $\left.\frac{\partial f}{\partial x}\right|_{(x,y,z)}$ or $f_x(x,y,z)$.

ii) If $\lim_{k\to 0}\frac{f(x,y+k,z)-f(x,y,z)}{k}$ exists finitely, then it is called the ***first order partial derivative*** with respect to y and is denoted by $\left.\frac{\partial f}{\partial y}\right|_{(x,y,z)}$ or $f_y(x,y,z)$.

iii) If $\lim_{l\to 0}\frac{f(x,y,z+l)-f(x,y,z)}{l}$ exists finitely, then it is called the ***first order partial derivative*** with respect to z and is denoted by $\left.\frac{\partial f}{\partial z}\right|_{(x,y,z)}$ or $f_z(x,y,z)$.

4.1.3 Examples

1. Find all the first order partial derivatives of

$$\boldsymbol{f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}}$$

→ Let $x^2+y^2 \neq 0$. Then

$$f_x(x,y) = \frac{y}{x^2+y^2} - \frac{2x^2y}{(x^2+y^2)^2} = \frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

$$f_x(x,y) = \frac{x}{x^2+y^2} - \frac{2xy^2}{(x^2+y^2)^2} = \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

Now when $x^2+y^2=0$, i.e. when $x=y=0$, then

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

and
$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k}$$
$$= \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$
$$= \lim_{k\to 0} \frac{0-0}{k}$$
$$= 0$$

Therefore

$$f_x(x,y) = \begin{cases} \dfrac{y(y^2 - x^2)}{(x^2+y^2)^2}, & x^2+y^2 \neq 0 \\ 0 \quad , & x^2+y^2 = 0 \end{cases}$$

and

$$f_y(x,y) = \begin{cases} \dfrac{x(x^2 - y^2)}{(x^2+y^2)^2}, & x^2+y^2 \neq 0 \\ 0 \quad , & x^2+y^2 = 0 \end{cases}$$

2. Show that neither $f_x(0,0)$ nor $f_y(0,0)$ exists, where

$$f(x,y) = \begin{cases} x\sin\dfrac{1}{x} + y\sin\dfrac{1}{y}, & xy \neq 0 \\ x\sin\dfrac{1}{x}, & x \neq 0, y = 0 \\ y\sin\dfrac{1}{y}, & x = 0, y \neq 0 \\ 0, & x = 0, y = 0 \end{cases}$$

→ Here

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h \sin \frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} \sin \frac{1}{h}$$

Since $\lim_{h \to 0} \sin(1/h)$ does not exist, therefore $f_x(0,0)$ does not exist.

Similarly since $\lim_{k \to 0} \sin(1/k)$ does not exist, therefore $f_y(0,0)$ does not exist.

3. If $f(x,y) = |x| + |y|, (x,y) \in \mathbb{R}^2$, then show that neither $f_x(0,0)$ nor $f_y(0,0)$ exist.

→ Here

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{|h| - 0}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

Since $\lim_{h \to 0} (|h|/h)$ does not exist, therefore $f_x(0,0)$ does not exist.

Similarly since

$$f_y(0,0) = \lim_{k \to 0} \frac{|k|}{k}$$

does not exist, therefore $f_y(0,0)$ does not exist.

4.2 Higher order partial derivatives

Let $D \subseteq \mathbb{R}^2$. Let $f: D \to \mathbb{R}$ be a function and (x,y) be an interior point of D. We define the second order partial derivatives as follows:

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = \lim_{h \to 0} \frac{f_x(x+h,y) - f_x(x,y)}{h}$$

$$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = \lim_{k \to 0} \frac{f_y(x,y+k) - f_y(x,y)}{k}$$

$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \lim_{h\to 0}\frac{f_y(x+h,y) - f_y(x,y)}{h}$$

$$f_{yx}(x,y) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \lim_{k\to 0}\frac{f_x(x,y+k) - f_x(x,y)}{k}$$

provided all limits exist.

- **Note:** $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are called ***mixed order partial derivatives***.

4.2.1 Examples

1. If

$$\boldsymbol{f(x,y) = \begin{cases} x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right), & (x,y) \neq 0 \\ 0 \quad , & (x,y) = 0 \end{cases}}$$

show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

→ We know that

$$f_{xy}(0,0) = \lim_{h\to 0}\frac{f_y(h,0) - f_y(0,0)}{h} \quad \text{and} \quad f_{yx}(0,0)$$

$$= \lim_{k\to 0}\frac{f_x(0,k) - f_x(0,0)}{k}$$

provided the limits exist.

Now

$$f_y(h,0) = \lim_{k\to 0}\frac{f(h,k) - f(h,0)}{k} \qquad \text{[provided the limit exists]}$$

$$= \lim_{k\to 0}\frac{h^2 \tan^{-1}\left(\frac{k}{h}\right) - k^2 \tan^{-1}\left(\frac{h}{k}\right)}{k}$$

$$= \lim_{k\to 0}\frac{h^2 \tan^{-1}\left(\frac{k}{h}\right)}{k} - \lim_{k\to 0} k \tan^{-1}\left(\frac{h}{k}\right)$$

$$= \lim_{k\to 0}\frac{h^2 \tan^{-1}\left(\frac{k}{h}\right)}{k} \quad \left[\begin{array}{l} \because \tan^{-1}\left(\frac{h}{k}\right) \text{ is bdd on } N'(0,\delta) \text{ and } \lim\limits_{k\to 0} k = 0, \\ \therefore \lim\limits_{k\to 0} k \tan^{-1}\left(\frac{h}{k}\right) = 0 \end{array}\right]$$

$$= \lim_{k\to 0} \frac{h^2}{\left(1+\frac{k^2}{h^2}\right)} \times \frac{1}{h} \left[\frac{0}{0} \text{ form, using L'Hospital's rule}\right]$$

$$= h$$

and

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} \qquad \text{[provided the limit exists]}$$

$$= \lim_{k\to 0} \frac{0-0}{k}$$

$$= 0$$

Therefore

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1 \tag{1}$$

Again

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} \qquad \text{[provided the limit exists]}$$

$$= \lim_{h\to 0} \frac{h^2 \tan^{-1}\left(\frac{k}{h}\right) - k^2 \tan^{-1}\left(\frac{h}{k}\right)}{h}$$

$$= \lim_{h\to 0} h \tan^{-1}\left(\frac{k}{h}\right) - \lim_{h\to 0} \frac{k^2 \tan^{-1}\left(\frac{h}{k}\right)}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \tan^{-1}\left(\frac{k}{h}\right)}{k} \begin{bmatrix} \because \tan^{-1}\left(\frac{k}{h}\right) \text{ is bdd on } N'(0,\delta) \text{ and } \lim\limits_{h\to 0} h = 0, \\ \therefore \lim\limits_{h\to 0} h \tan^{-1}\left(\frac{k}{h}\right) = 0 \end{bmatrix}$$

$$= -\lim_{h\to 0} \frac{k^2}{\left(1+\frac{h^2}{k^2}\right)} \times \frac{1}{k} \left[\frac{0}{0} \text{ form, using L'Hospital's rule}\right]$$

$$= -k$$

and

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \qquad \text{[provided the limit exists]}$$

$$= \lim_{h\to 0} \frac{0-0}{h}$$

$$= 0$$

Therefore

$$f_{yx}(0,0) = \lim_{k \to 0} \frac{-k - 0}{k} = -1 \qquad (2)$$

From (1) and (2), we see that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

2. If

$$\boldsymbol{f(x,y)} = \begin{cases} \boldsymbol{xy}, & |\boldsymbol{x}| \geq |\boldsymbol{y}| \\ \\ -\boldsymbol{xy}, & |\boldsymbol{x}| < |\boldsymbol{y}| \end{cases}$$

show that $\boldsymbol{f_{xy}(0,0) \neq f_{yx}(0,0)}$.

→ We know that

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h} \quad \text{and} \quad f_{yx}(0,0)$$

$$= \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

provided the limits exist.

Now

$$f_y(h,0) = \lim_{k \to 0} \frac{f(h,k) - f(h,0)}{k} \qquad \text{[provided the limit exists]}$$

$$= \lim_{k \to 0} \frac{hk}{k} \qquad [\because k \to 0, |h| > |k|]$$

$$= h$$

and

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} \qquad \text{[provided the limit exists]}$$

$$= \lim_{k \to 0} \frac{0 - 0}{k}$$

$$= 0$$

Therefore

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{h - 0}{h} = 1 \qquad (1)$$

Again

$$f_x(0,k) = \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h} \qquad \text{[provided the limit exists]}$$

$$= \lim_{h \to 0} \frac{-hk}{h} \qquad [\because h \to 0, |h| < |k|]$$
$$= -k$$

and

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} \qquad \text{[provided the limit exists]}$$
$$= \lim_{h \to 0} \frac{0 - 0}{h}$$
$$= 0$$

Therefore

$$f_{yx}(0,0) = \lim_{k \to 0} \frac{-k - 0}{k} = -1 \qquad (2)$$

From (1) and (2), we see that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

3. If

$$\boldsymbol{f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 \neq 0 \end{cases}}$$

show that $\boldsymbol{f_{xy}(0,0) \neq f_{yx}(0,0)}$.

4. If

$$\boldsymbol{f(x,y) = \begin{cases} \dfrac{x^2 y^2}{x^2 + y^2}, & (x,y) \neq 0 \\ 0, & (x,y) \neq 0 \end{cases}}$$

show that $\boldsymbol{f_{xy}(0,0) = f_{yx}(0,0)}$.

5. If

$$\boldsymbol{f(x,y) = \begin{cases} (x^2 + y^2)\log(x^2 + y^2), & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}}$$

show that $\boldsymbol{f_{xy}(0,0) = f_{yx}(0,0)}$.

1.5. Sufficient condition for continuity

Let $D \subseteq \mathbb{R}^2, f: D \to \mathbb{R}$ be a function and $(a,b) \in D$. If

(i) $f_x(a,b)$ $[$or $f_y(a,b)]$ exists

(ii) $f_y(x,y)$ [or $f_x(x,y)$] is bounded in some neighbourhood of (a,b)

Then f is continuous at (a,b).

Proof: Since f_y is bounded in some nbd of (a,b), therefore there exists a nbd N of (a,b) in which f_y is bounded.

We choose h,k in a way such that $(a+h,b),(a,b+k),(a+h,b+k) \in N$.

Now

$$f(a+h,b+k)-f(a,b)$$
$$= f(a+h,b+k)-f(a+h,b)+f(a+h,b)-f(a,b) \qquad (1)$$

Since $f_x(a,b)$ exists, therefore

$$\lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h} = f_x(a,b)$$
$$\Longrightarrow f(a+h,b)-f(a,b) = h\{f_x(a,b)+\epsilon(h)\} \qquad (2)$$

where ϵ is a function of h such that $\epsilon \to 0$ as $h \to 0$.

Let $t(y) = f(a+h,y)$. Then $t'(y) = f_y(a+h,y)$.

Since f_y exists in the nbd of (a,b), therefore $t'(y)$ exists in $[b,b+k]$.

Therefore by Lagrange's mean value theorem of differential calculus, there exists $\theta \in (0,1)$ such that

$$t(b+k)-t(b) = kt'(b+\theta k)$$
$$\Longrightarrow f(a+h,b+k)-f(a+h,b) = kf_y(a+h,b+\theta k) \qquad (3)$$

From $(1),(2)$ and (3), we get

$$f(a+h,b+k)-f(a,b)$$
$$= kf_y(a+h,b+\theta k)+h\{f_x(a,b)+\epsilon(h)\} \qquad (4)$$

Since f_y is bounded in N and $\lim_{k\to 0} k = 0$, therefore $\lim_{k\to 0} kf_y(a+h,b+\theta k) = 0$

$\therefore$ From (4),

$$\lim_{(h,k)\to(0,0)} f(a+h,b+k)-f(a,b) = 0$$

$$\Rightarrow \lim_{(h,k)\to(0,0)} f(a+h,b+k) = f(a,b)$$

This shows that f is continuous at (a,b).

- **Note:** The above condition is not necessary. For example, consider the function

$$f(x,y) = \begin{cases} |x|+|y|, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Let $\epsilon > 0$ be arbitrary. Then

$$|f(x,y) - f(0,0)| = \big||x|+|y|\big| = |x|+|y| < 2\sqrt{x^2+y^2} < \epsilon$$

holds if $\sqrt{x^2+y^2} < \delta = \epsilon/2$.

Therefore $f(x,y)$ is continuous at $(0,0)$ whenever $\sqrt{x^2+y^2} < \delta$ holds.

But

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

which does not exist. Similarly we can show that $f_y(0,0)$ does not exist.

- **Theorem 5.1: Let $D \subseteq \mathbb{R}^2$, $f: D \to \mathbb{R}$ be a function, f_x and f_y both exist and be bounded in D. Then f is continuous in D.**

Proof: Let $(a,b) \in D$.

We choose h and k such that $(a+h,b), (a,b+k), (a+h,b+k) \in D$.

Now

$$f(a+h,b+k) - f(a,b)$$
$$= f(a+h,b+k) - f(a+h,b) + f(a+h,b) - f(a,b) \tag{1}$$

Let $t_1(y) = f(a+h,y)$ and $t_2(x) = f(x,b)$.

Then $t_1'(y) = f_y(a+h,y)$ and $t_2'(x) = f_x(x,b)$.

Since f_y and f_x exist in the D, therefore $t_1'(y)$ and $t_2'(x)$ both exist in $[b,b+k]$ and $[a,a+h]$ respectively.

Therefore by Lagrange's mean value theorem of differential calculus, there exists $\theta_1, \theta_2 \in (0,1)$ such that

$$t_1(b+k) - t_1(b) = kt_1'(b+\theta_1 k)$$
$$\Rightarrow f(a+h,b+k) - f(a+h,b) = kf_y(a+h,b+\theta_1 k) \tag{2}$$

and

$$t_2(a+h) - t_2(a) = ht_2'(a+\theta_2 h)$$

$$\Rightarrow f(a+h,b) - f(a+h,b) = hf_x(a+\theta_2 h, b) \quad (3)$$

From (1), (2) and (3), we get

$$f(a+h,b+k) - f(a,b) = kf_y(a+h,b+\theta_1 k) + hf_x(a+\theta_2 h,b) \quad (4)$$

Since f_x and f_y are bounded in D and $\lim_{h\to 0} h = \lim_{k\to 0} k = 0$, therefore $\lim_{k\to 0} kf_y(a+h,b+\theta k) = 0$ and $\lim_{k\to 0} hf_x(a+\theta_2 h,b) = 0$.

∴ From (4),

$$\lim_{(h,k)\to(0,0)} f(a+h,b+k) - f(a,b) = 0$$

$$\Rightarrow \lim_{(h,k)\to(0,0)} f(a+h,b+k) = f(a,b)$$

This shows that f is continuous at (a,b).

1.6. Differentiability

Let $D \subseteq \mathbb{R}^2, f: D \to \mathbb{R}$ be a function and (a,b) be an interior point of D. f is said to be ***differentiable*** at (a,b) if for $(a+h,b+k) \in D$

$$f(a+h,b+k) - f(a,b) = Ah + Bk + h\epsilon(h,k) + k\eta(h,k)$$

where A and B are independent of h and k and ϵ,η are functions of h and k such that $(\epsilon,\eta) \to (0,0)$ as $(h,k) \to (0,0)$.

6.1 Necessary condition for differentiability

Let D be an open subset of $\mathbb{R}^2$, $f: D \to \mathbb{R}$ be a function and (a,b) be an interior point of D. If f is differentiable at (a,b), then

(i) $f_x(a,b)$ and $f_y(a,b)$ exist

(ii) f is continuous at (a,b).

Proof: Since f is differentiable at (a,b), so for $(a+h,b+k) \in D$

$$f(a+h,b+k) - f(a,b) = Ah + Bk + h\epsilon(h,k) + k\eta(h,k) \quad (1)$$

where A and B are independent of h and k and ϵ, η are functions of h and k such that $(\epsilon, \eta) \to (0,0)$ as $(h, k) \to (0,0)$.

We take $h \neq 0, k = 0$. Then (1) becomes

$$f(a+h, b) - f(a, b) = Ah + h\epsilon = h(A + \epsilon)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(a+h, b+k) - f(a, b)}{h} = A \qquad [\because \epsilon \to 0 \text{ as } h \to 0]$$

$$\Rightarrow f_x(a, b) = A$$

Similarly taking $h = 0, k \neq 0$, we obtain $f_y(a, b) = B$.

Again from (1),

$$\lim_{(h,k) \to (0,0)} \{f(a+h, b+k) - f(a, b)\} = 0$$

$$\Rightarrow \lim_{(h,k) \to (0,0)} f(a+h, b+k) = f(a, b)$$

This shows that f is continuous at (a, b).

- **Note:** The above condition is not sufficient, i.e. if f_x and f_y exist for a function f at a point $(a, b) \in D$ and f is continuous at (a, b), then the function may not be differentiable at (a, b). For example, consider

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases} \qquad (1)$$

Now

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h, 0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1 \qquad (2)$$

and

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0, k) - f(0,0)}{h} = \lim_{h \to 0} \frac{-k}{k} = -1 \qquad (3)$$

Therefore $f_x(0,0)$ and $f_y(0,0)$ both exist.

Also f is continuous at $(0,0)$ **[exercise 3.3, problem 1.(i)]**.

We now check whether f is differentiable at $(0,0)$.

If possible, let f be differentiable at $(0,0)$. Then we may write

$$f(h, k) - f(0,0) = hf_x(0,0) + kf_y(0,0) + h\epsilon + k\eta$$

$$\Rightarrow f(h,k) - f(0,0) = h\{f_x(0,0) + \epsilon\} + k\{f_y(0,0) + \eta\} \qquad (4)$$

where ϵ, η are functions of h and k such that $(\epsilon, \eta) \to (0,0)$ as $(h,k) \to (0,0)$.

Thus from (1), (2), (3) and (4), we obtain

$$\frac{h^3 - k^3}{h^2 + k^2} = h\{1 + \epsilon\} + k\{-1 + \eta\} \qquad (5)$$

Let $(h,k) \to (0,0)$ along the path $k = mh$. Then from (5), we have

$$\frac{h^3(1 - m^3)}{h^2(1 + m^2)} = h\{1 + \epsilon + m(-1 + \eta)\}$$

$$\Rightarrow \frac{(1 - m^3)}{(1 + m^2)} = 1 - m + \epsilon + m\eta \qquad (6)$$

As $(h,k) \to (0,0)$, the L.H.S of (6) tends to $(1 - m^3)/(1 + m^2)$ but the R.H.S tends to $1 - m$.

Thus L.H.S ≠ R.H.S, in general. Hence f is not differentiable at $(0,0)$.

6.1.1 Exercise

1. Show that

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is not differentiable at $(0,0)$.

→ Similar to the problem in previous note.

2. Show that $f(x,y) = \sqrt{|xy|}$ is not differentiable at $(0,0)$.

→ If possible, let f be differentiable at $(0,0)$. Then we may write

$$f(h,k) - f(0,0)$$
$$= h(f_x(0,0) + \epsilon) + k(f_y(0,0) + \eta) \qquad (1)$$

where ϵ and η are functions of h and k and $(\epsilon, \eta) \to (0,0)$ as $(h,k) \to (0,0)$.

Now

$$\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$

and

$$\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0-0}{k}=0$$

Therefore from (1),

$$f(h,k)-f(0,0)=h\epsilon+k\eta$$

$$\Rightarrow \left|\sqrt{hk}\right|=h\epsilon+k\eta$$

Let $(h,k)\to(0,0)$ along the path $k=mh$. Then the above equation becomes

$$h\sqrt{m}=h(\epsilon+m\eta)$$

$$\Rightarrow \sqrt{m}=(\epsilon+m\eta)$$

as $(h,k)\to(0,0)$, L.H.S $\to\sqrt{m}$ and R.H.S $\to 0$. ∴ L.H.S ≠ R.H.S, in general.

Hence f is not differentiable at $(0,0)$.

3. Show that

$$\boldsymbol{f(x,y)=\begin{cases} x\sin\left(4\tan^{-1}\frac{y}{x}\right), & x>0\\ 0 \quad , & x=0\end{cases}}$$

is not differentiable at $(0,0)$.

4. Show that

$$\boldsymbol{f(x,y)=\begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0 \quad , & (x,y)=(0,0)\end{cases}}$$

is differentiable at $(0,0)$.

→ Here

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$

and

$$\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0-0}{k}=0$$

Now

$$f(h,k)-f(0,0)=hk\frac{h^2-k^2}{h^2+k^2}=h\frac{h^2k}{h^2+k^2}-k\frac{hk^2}{h^2+k^2}$$

where $(h,k)\neq(0,0)$.

Let $\epsilon(h,k)=\frac{h^2k}{h^2+k^2}$ and $\eta(h,k)=-\frac{hk^2}{h^2+k^2}$

Then $(\epsilon,\eta)\to(0,0)$ as $(h,k)\to(0,0)$.

Thus

$$f(h,k)-f(0,0)=h\{f_x(0,0)+\epsilon\}+k\{f_y(0,0)+\eta\}$$

where ϵ and η are functions of h and k and $(\epsilon,\eta)\to(0,0)$ as $(h,k)\to(0,0)$.

Hence f is differentiable at $(0,0)$.

5. Show that

$$\boldsymbol{f(x,y)=\begin{cases}\dfrac{x^6-2y^4}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}}$$

is differentiable at $(0,0)$.

[Hint: Here $f_x(0,0)=f_y(0,0)=0$ and $(h,k)=h^5/(h^2+k^2)$, $\eta(h,k)=-2k^3/(h^2+k^2)$]

6. Show that

$$\boldsymbol{f(x,y)=\begin{cases}x, & |y|<|x|\\ -x, & |y|\geq|x|\end{cases}}$$

is not differentiable at $(0,0)$.

7. Let $f(x,y)=g\left(\sqrt{x^2+y^2}\right)$ where

$$\boldsymbol{g(z)=\begin{cases}z^2\sin\dfrac{1}{z}, & z\neq 0\\ 0, & z=0\end{cases}}$$

Show that f is differentiable at $(0,0)$ but f_x and f_y are not continuous at $(0,0)$.

→ Here

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right), & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

Now

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\frac{1}{|h|}}{h} = \lim_{h\to 0} h \sin\frac{1}{|h|} = 0$$

and

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{k^2 \sin\frac{1}{|k|}}{k} = \lim_{k\to 0} k \sin\frac{1}{|k|} = 0$$

Therefore

$$\begin{aligned} &f(h,k) - f(0,0) \\ &= (h^2 + k^2)\sin\left(\frac{1}{\sqrt{h^2 + k^2}}\right) \\ &= h\left\{h \sin\left(\frac{1}{\sqrt{h^2 + k^2}}\right)\right\} + k\left\{k \sin\left(\frac{1}{\sqrt{h^2 + k^2}}\right)\right\} \end{aligned}$$

where $h^2 + k^2 \neq 0$.

Let $\epsilon(h,k) = h\sin\left(\frac{1}{\sqrt{h^2+k^2}}\right)$ and $\eta(h,k) = k\sin\left(\frac{1}{\sqrt{h^2+k^2}}\right)$

Then $(\epsilon,\eta) \to (0,0)$ as $(h,k) \to (0,0)$.

Thus

$$f(h,k) - f(0,0) = h\{f_x(0,0) + \epsilon\} + k\{f_y(0,0) + \eta\}$$

where ϵ and η are functions of h and k and $(\epsilon,\eta) \to (0,0)$ as $(h,k) \to (0,0)$.

Hence f is differentiable at $(0,0)$.

Now

$$f_x(x,y) = \begin{cases} 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$$

Let $(x,y) \to (0,0)$ along the path $y = mx$, where $x > 0$. Therefore

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{x\to 0}\left[2x\sin\left(\frac{1}{x\sqrt{1+m^2}}\right) - \frac{x}{x\sqrt{1+m^2}}\cos\frac{1}{x\sqrt{1+m^2}}\right]$$

As $\lim_{x\to 0}\cos\frac{1}{x\sqrt{1+m^2}}$ does not exist, therefore $\lim_{(x,y)\to(0,0)} f_x(x,y)$ does not exist and consequently, f_x is not continuous at $(0,0)$.

Again

$$f_y(x,y) = \begin{cases} 2y\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{y}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0 \quad , & x^2+y^2 = 0 \end{cases}$$

Due to symmetry, f_y is not continuous at $(0,0)$.

6.2 Sufficient condition for differentiability

Let D be an open subset of $\mathbb{R}^2$, $f: D \to \mathbb{R}$ be a function and (a,b) be an interior point of D. If

(i) $f_x(a,b)$ exists (or $f_y(a,b)$ exists)

(ii) f_y is continuous at (a,b) (or f_x is continuous at (a,b))

Then f is differentiable at (a,b).

Proof: Since f_y is continuous at (a,b), so there exists a nbd N of (a,b) in which both f and f_y are defined.

We choose $(h,k) \neq (0,0)$ such that $(a+h,b), (a+h,b+k) \in N$. Now

$$f(a+h,b+k) - f(a,b)$$
$$= f(a+h,b+k) - f(a+h,b) + f(a+h,k) - f(a,b) \qquad (1)$$

Since $f_x(a,b)$ exists, therefore

$$\lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} = f_x(a,b)$$
$$\Rightarrow f(a+h,b) - f(a,b) = h\{f_x(a,b) + \epsilon\} \qquad (2)$$

where ϵ is a function of h and $\epsilon \to 0$ as $h \to 0$.

Let $t(y) = f(a+h,y)$. Then $t'(y) = f_y(a+h,y)$.

Since f_y exists in N, so $t'(y)$ exists in $[b, b+k]$.

By Lagrange's MVT of differential calculus, there exists $\theta \in (0,1)$ such that

$$t(b+k) - t(b) = kt'(b+\theta k)$$
$$\Rightarrow f(a+h,b+k) - f(a+h,b) = kf_y(a+h,b+\theta k) \qquad (3)$$

From $(1), (2)$ and (3), we get

$$f(a+h,b+k) - f(a,b) = kf_y(a+h,b+\theta k) + h\{f_x(a,b) + \epsilon\} \qquad (4)$$

Since f_y is continuous at (a, b), so

$$\lim_{(h,k)\to(0,0)} f_y(a+h, b+\theta k) = f_y(a, b)$$

$$\implies f_y(a+h, b+\theta k) = f_y(a, b) + \eta$$

where η is a function of (h, k) and $\eta \to 0$ as $(h, k) \to (0,0)$.

Thus from (4), we have

$$f(a+h, b+k) - f(a, b) = h\{f_x(a, b) + \epsilon\} + k\{f_y(a, b) + \eta\}$$

where ϵ and η are functions of h and k and $(\epsilon, \eta) \to (0,0)$ as $(h, k) \to (0,0)$.

Hence f is differentiable at (a, b).

- **Note:** The converse of the theorem is not true. Let us consider the function

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y}, & (x, y) \neq (0,0) \\ x^2 \sin\frac{1}{x}, & x \neq 0, y = 0 \\ y^2 \sin\frac{1}{y}, & x = 0, y \neq 0 \\ 0 & (x, y) = (0,0) \end{cases}$$

Now

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h, 0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h}}{h} = \lim_{h\to 0} h \sin\frac{1}{h} = 0$$

and

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0, k) - f(0,0)}{k} = \lim_{k\to 0} \frac{k^2 \sin\frac{1}{k}}{k} = \lim_{k\to 0} k \sin\frac{1}{k} = 0$$

Therefore

$$f_x(0,0) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and

$$f_y(0,0) = \begin{cases} 2y \sin\frac{1}{y} - \cos\frac{1}{y}, & y \neq 0 \\ 0, & y = 0 \end{cases}$$

As $\lim_{x\to 0}\cos\frac{1}{x}$ and $\lim_{x\to 0}\cos\frac{1}{y}$ do not exist, therefore neither $\lim_{(x,y)\to(0,0)} f_x(x,y)$ nor $\lim_{(x,y)\to(0,0)} f_y(x,y)$ exists. Hence f_x and f_y are not continuous at $(0,0)$.

Now

$$f(0+h,0+k)-f(0,0)=h^2\sin\frac{1}{h}+k^2\sin\frac{1}{k}$$

where $(h,k)\neq(0,0)$.

Let $\epsilon = h\sin 1/h$ and $\eta = k\sin 1/k$. Then ϵ and η are functions of h and k and $(\epsilon,\eta)\to(0,0)$ as $(h,k)\to(0,0)$.

Thus $f(h,k)-f(0,0)=h\{f_x(0,0)+\epsilon\}+k\{f_y(0,0)+\eta\}$ where ϵ and η are functions of h and k and $(\epsilon,\eta)\to(0,0)$ as $(h,k)\to(0,0)$.

Hence f is differentiable at $(0,0)$.

6.2.1 Exercise

1. Let

$$\boldsymbol{f(x)=\begin{cases} x^2\sin\frac{1}{x}+y^2\sin\frac{1}{y}, & (x,y)\neq(0,0)\\ x^2\sin\frac{1}{x}, & x\neq 0, y=0\\ y^2\sin\frac{1}{y}, & x=0, y\neq 0\\ 0 & (x,y)=(0,0)\end{cases}}$$

Show that f does not satisfy the sufficient condition of differentiability although f is differentiable at $(0,0)$.

2. Show that

$$\boldsymbol{f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0\ , & (x,y)=(0,0)\end{cases}}$$

satisfies the sufficient condition of differentiability.

[Hint: Show that f is continuous at $(0,0)$ and $f_x(0,0)$ and $f_y(0,0)$ both exist]

Chapter 2

Matrices

2: Introduction of Matrices

2.1 Definition 1:

A rectangular arrangement of mn numbers, in m rows and n columns and enclosed within a bracket is called a matrix. We shall denote matrices by capital letters as *A*,*B*, *C* etc.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{m \times n}$$

A is a matrix of order *m by n.*

Remark: A matrix is not just a collection of elements but every element has assigned a definite position in a particular row and column.

2.2 Special Types of Matrices:

1. **Square matrix**:

A matrix in which numbers of rows are equal to number of columns is called a square matrix.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 5 & -8 \\ 0 & -3 & -4 \\ 6 & 8 & 9 \end{pmatrix}$$

2. Diagonal matrix:

A square matrix *A* = $(a_{ij})_{n \times n}$ is called a diagonal matrix if each of its non-diagonal element is zero.

That is $a_{ij} = 0 \; if \; i \neq j$ and at least one element $a_{ii} \neq 0$.

Example:

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

3. **Identity Matrix**

A diagonal matrix whose diagonal elements are equal to 1 is called identity matrix and denoted by I_n.

That is $a_{ij} = \begin{cases} 0 & if \; i \neq j \\ 1 & if \; i = j \end{cases}$

Example:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. **Upper Triangular matrix**:

A square matrix said to be a Upper triangular matrix if $a_{ij} = 0 \; if \; i > j$.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 8 \\ 0 & -2 & 5 \\ 0 & 0 & 7 \end{pmatrix}$$

5. **Lower Triangular Matrix**:

A square matrix said to be a Lower triangular matrix if $a_{ij} = 0 \; if \; i < j$.

Example:

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 & 0 \\ 7 & 0 & 0 \\ 9 & 6 & 2 \end{pmatrix}$$

6. **Symmetric Matrix**:

A square matrix *A* = $(a_{ij})_{n \times n}$ said to be a symmetric if $a_{ij} = a_{ji}$ for all *i* and *j.*

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 8 & -2 & 7 \\ -2 & -9 & 3 \\ 7 & 3 & 5 \end{pmatrix}$$

7. Skew- Symmetric Matrix:

A square matrix $A = (a_{ij})_{n\times n}$ said to be a skew-symmetric if $a_{ij} = -a_{ji}$ for all *i* and *j.*

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{22} & a_{23} \\ -a_{13} & -a_{23} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 8 & -2 & 7 \\ 2 & -9 & 3 \\ -7 & -3 & 5 \end{pmatrix}$$

8. Zero Matrix:

A matrix whose all elements are zero is called as Zero Matrix and order $n \times m$ Zero matrix denoted by $0_{n\times m}$.

Example:

$$0_{3\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

9. Row Vector

A matrix consists a single row is called as a row vector or row matrix.

Example:

$$A = (a_{11} \quad a_{12} \quad a_{13}) \qquad B = (7 \quad 4 \quad -3)$$

10. Column Vector

A matrix consists a single column is called a column vector or column matrix.

Example:

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \qquad B = \begin{pmatrix} 9 \\ -7 \\ 3 \end{pmatrix}$$

2.3: Matrix Algebra

2.3.1. **Equality of two matrices**:

Two matrices A and B are said to be equal if

(i) They are of same order.
(ii) Their corresponding elements are equal.

That is if $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ then $a_{ij} = b_{ij}$ for all *i* and *j.*

2.3.2. Scalar multiple of a matrix

Let k be a scalar then scalar product of matrix $A = (a_{ij})_{m\times n}$ given denoted by *kA* and given by $kA = (ka_{ij})_{m\times n}$ or

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ \vdots & & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

2.3.3. Addition of two matrices:

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ are two matrices with same order then sum of the two matrices are given by

$$A + B = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n}$$

Example 1: let

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 0 & 2 \\ -1 & 1 & 8 \end{pmatrix}.$$

Find (i) 5B (ii) A + B (iii) 4A – 2B (iv) 0 A

2.3.4. Multiplication of two matrices:

Two matrices *A* and *B* are said to be confirmable for product *AB* if number of columns in *A* equals to the number of rows in matrix *B*. Let *A* = be two $(a_{ij})_{m\times n}$ $and\ B = (b_{ij})_{n\times r}$ matrices the product matrix *C*= *AB*, is matrix of order $m\times r$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots\dots\dots\dots + a_{in}b_{nj}$$

Example 2: Let $A = \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -5 & -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ -5 & 0 \\ 6 & -2 \\ -1 & -3 \end{pmatrix}$

Calculate (i) AB (ii) BA

(iii) is AB = BA ?

2.3.5. Integral power of Matrices:

Let A be a square matrix of order n, and m be positive integer then we define

$A^m = A \times A \times A \dots\dots . \times A$ (m times multiplication)

2.3.6. Properties of the Matrices

Let A, B and C are three matrices and λ and μ are scalars then

(i) $A + (B + C) = (A + B) + C$ Associative Law

(ii) $\lambda(A+B) = \lambda A + \lambda B$ Distributive law

(iii) $\lambda(\mu A) = (\lambda\mu)A$ Associative Law

(iv) $(\lambda A)B = \lambda(AB)$ Associative Law

(v) $A(BC) = (AB)C$ Associative Law

(vi) $A(B+C) = AB + AC$ Distributive law

2.3.7. Transpose:

The transpose of matrix $A = (a_{ij})_{m\times n}$, written A^t (A' or A^T) is the matrix obtained by writing the rows of A in order as columns.

That is $A^t = (a_{ji})_{n\times m}$.

Properties of Transpose:

(i) $(A+B)^t = (A^t + B^t)$

(ii) $(A^t)^t = A$

(iii) $(kA)^t = k\,A^t$ for scalar k.

(iv) $(AB)^t = B^t A^t$

Example 3: Using the following matrices A and B, Verify the transpose properties

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 5 & -4 & 3 \\ 1 & -2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 6 & -2 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

Proof: (i) Let a_{ij} and b_{ij} are the $(i,j)^{th}$element of the matrix A and B respectively. Then $a_{ij}+b_{ij}$ is the $(i,j)^{th}$ element of matrix $A+B$ and it is $(j,i)^{th}$ element of the matrix $(A+B)^t$

Also a_{ij} and b_{ij} are the $(j,i)^{th}$element of the matrix A^t and B^trespectively. Therefore $a_{ij}+b_{ij}$ is the $(j,i)^{th}$ element of the matrix $A^t + B^t$

(ii) Let $(i,j)^{th}$ element of the matrix A is a_{ij}, it is $(j,i)^{th}$ element of the A^t then it is $(i,j)^{th}$ element of the matrix $(A^t)^t$

(iii) try

(iv) $c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots\dots\dots + a_{in}b_{nj}$ is the $(i,k)^{th}$ element of the AB It is result of the multiplication of the i^{th} row and k^{th} column and it is $(k,i)^{th}$ element of the matrix $(AB)^t$.

$B^t A^t$, $(k,i)^{th}$ element is the multiplication of k^{th} row of $B^t$with i^{th} column of A^t, That is k^{th} column of B with i^{th} row of A.

2.3.8 A square matrix A is said to be symmetric if $A = A^t$.

Example 4:

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -4 & -2 \\ 1 & -2 & -3 \end{pmatrix}$$, A is symmetric by the definition of symmetric matrix.

Then

$$A^t = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -4 & -2 \\ 1 & -2 & -3 \end{pmatrix}$$

That is $A = A^t$

2.3.9 A square matrix A is said to be skew- symmetric if $A = -A^t$

Example 5:

$$A = \begin{pmatrix} 1 & 3 & -1 \\ -3 & -5 & 8 \\ 1 & 8 & 9 \end{pmatrix}$$

(i) AA^t and A^tA are both symmetric.

(ii) $A + A^t$ is a symmetric matrix.

(iii) $A - A^t$ is a skew-symmetric matrix.

(iv) If A is a symmetric matrix and m is any positive integer then A^m is also symmetric.

(v) If A is skew symmetric matrix then odd integral powers of A is skew symmetric, while positive even integral powers of A is symmetric.

If *A* and *B* are symmetric matrices then

(vi) $(AB + BA)$ is symmetric.

(vii) $(AB - BA)$ is skew-symmetric.

Exercise 1: Verify the (i) , (ii) and (iii) using the following matrix A.

$$A = \begin{pmatrix} 1 & 3 & 5 \\ -3 & -5 & 10 \\ 1 & 8 & 9 \end{pmatrix}$$

2.4: Determinant, Minor and Adjoint Matrices

Definition 2.4.1:

Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n , then the number $|A|$ called determinant of the matrix A.

(i) Determinant of 2 $\times$ 2 matrix

Let A= $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then $|A|$ = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ = $a_{11}a_{22} - a_{12}a_{21}$

(ii) Determinant of 3 × 3 matrix

Let B = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Then $|B|$ = $a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$$|B| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) - a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Exercise 2: Calculate the determinants of the following matrices

(i) $A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 1 & 9 & 5 \end{pmatrix}$ (ii) $B = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{pmatrix}$

2.4.2 Properties of the Determinant:

a. The determinant of a matrix A and its transpose A^t are equal.

$$|A| = |A^t|$$

b. Let A be a square matrix

(i) If A has a row (column) of zeros then $|A| = 0$.

(ii) If A has two identical rows (or columns) then $|A| = 0$.

c. If A is triangular matrix then $|A|$ is product of the diagonal elements.

d. If A is a square matrix of order n and k is a scalar then $|kA| = k^n|A|$

2.4.3 Singular Matrix

If *A* is square matrix of order n, the *A* is called singular matrix when $|A| = 0$ and non-singular otherwise.

2.4.4 Minor and Cofactors:

Let A = $(a_{ij})_{n\times n}$ is a square matrix. Then M_{ij} denote a sub matrix of A with order (n-1) × (n-1) obtained by deleting its i^{th}row and j^{th} column. The determinant $|M_{ij}|$ is called the minor of the element a_{ij} of A.

The cofactor of a_{ij} denoted by A_{ij} and is equal to$(-1)^{i+j}|M_{ij}|$.

Exercise 3: Let $A = \begin{pmatrix} 5 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & -2 & -1 \end{pmatrix}$

(i) Compute determinant of A.

(ii) Find the cofactor matrix.

2.4.5 **Adjoin Matrix**:

The transpose of the matrix of cofactors of the element a_{ij} of A denoted by $adj\ A$ is called adjoin of matrix A.

Example 6: Find the adjoin matrix of the above example.

Theorem 2.4.1:

For any square matrix A,

$$A\,(adj\ A) = (adj\ A)A = |A|\ I$$ where I is the identity matrix of same order.

Proof: Let A = $\left(a_{ij}\right)_{n\times n}$

Since A is a square matrix of order n, then $adj\ A$ also in same order.

Consider

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots & \dots & \dots\dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$ then

$$adj\ A = \begin{pmatrix} A_{11} & A_{12} & \dots\dots & A_{1n} \\ A_{21} & A_{22} & \dots\dots & A_{2n} \\ \dots & \dots & \dots\dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}$$

Now consider the product $A\,(adj\ A)$

$$A\,(adj\ A) = \begin{pmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots & \dots & \dots\dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} & \dots\dots & A_{1n} \\ A_{21} & A_{22} & \dots\dots & A_{2n} \\ \dots & \dots & \dots\dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{n} a_{1j}\,A_{1j} & \sum_{j=1}^{n} a_{1j}\,A_{2j} & \dots\dots & \sum_{j=1}^{n} a_{1j}\,A_{nj} \\ \sum_{j=1}^{n} a_{2j}\,A_{1j} & \sum_{j=1}^{n} a_{2j}\,A_{2j} & \dots\dots & \sum_{j=1}^{n} a_{2j}\,A_{nj} \\ \dots\dots & \dots\dots & \dots\dots & \dots \\ \sum_{j=1}^{n} a_{nj}\,A_{1j} & \sum_{j=1}^{n} a_{nj}\,A_{2j} & \dots\dots & \sum_{j=1}^{n} a_{nj}\,A_{nj} \end{pmatrix}$$

$$= \begin{pmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots\dots & |A| \end{pmatrix}$$

(as we know that $\sum_{j=1}^{n} a_{ij}\,A_{ij} = |A|$ and $\sum_{j=1}^{n} a_{ij}\,A_{kj} = 0$ when $i \neq k$)

$$= |A| \begin{pmatrix} 1 & 0 & \dots.. & 0 \\ 0 & 1 & \dots.. & 0 \\ 0 & 0 & \dots.. & 0 \\ 0 & 0 & \dots\dots & 1 \end{pmatrix}$$

$= |A|\, I_n$ Where I_n is unit matrix of order n.

Theorem 2.4.2: If A is a non-singular matrix of order n, then $|adj\ A| = |A|^{n-1}$.

Proof: By the theorem 1

$$A\,(adj\ A) = |A|\ I$$

$$|A\,(adj\ A)| = \big||A|\ I\big|$$

$$|A||\,adj\ A\,| = |A|^n$$

$$|\,adj\ A\,| = |A|^{n-1}$$

Theorem 2.4.3: If A and B are two square matrices of order n then

$$adj(AB) = (adj\ B)(adj\ A)$$

Proof: By the theorem 1 $A\,(adj\ A) = |A|\ I$

Therefore $(AB)\ adj\ (AB) = adj\ (AB)AB = |AB|I$

Consider $(AB)(adj\ \ B\ adj\ A)$,

$$(AB)(adj\ \ B\ adj\ A) = A\,(B\ adj\ B)\ adj\ \ A$$

$$= A(|B|\ I)\ adj\ A$$

$$= |B|\ (A\ adj\ A)$$

$$= |B|\ \ |A|\ I$$

$$= |A||B|\ \ I$$

$$= |AB|I \qquad \dots\dots\dots\dots\dots\dots \text{(i)}$$

Also consider $(adj\ B\ \ .adj\ A)A\,B$

$$(adj\ B\ \ .adj\ A)A\,B = adj\ B\ (adj\ A\ A)B$$

$$= adj\ B\ |A|\ I\ B$$

$$= |A| adj\ B\ B$$

$$= |A||B|\ \ I$$

$$= |AB|I \qquad \dots\dots\dots\dots\dots\dots \text{(ii)}$$

Therefore from (i) and (ii) we conclude that

$$adj(AB) = (adj\ A)(adj\ B)$$

Some results of adjoint

(i) For any square matrix A $(adj\ A)^t = adj\ A^t$
(ii) The adjoint of an identity matrix is the identity matrix.
(iii) The adjoint of a symmetric matrix is a symmetric matrix.

.2.5: Inverse of a Matrix and Elementary Row Operations

2.5.1 Inverse of a Matrix

Definition 5.1:

If A and B are two matrices such that $AB = BA = I$, then each is said to be inverse of the other. The inverse of A is denoted by A^{-1}.

Theorem 2.5.1: (Existence of the Inverse)

The necessary and sufficient condition for a square matrix A to have an inverse is that $|A| \neq 0$ (That is A is non singular).

Proof: (i) The necessary condition

Let A be a square matrix of order n and B is inverse of it, then

$$AB = I$$

$$|AB| = |A||B| = I$$

Therefore $|A| \neq 0$.

(ii) The sufficient condition:

If $|A| \neq 0$, the we define the matrix B such that

$$B = \frac{1}{|A|}\ (adj\ A)$$

Then $AB = A\ \frac{1}{|A|}\ (adj\ A) = \frac{1}{|A|}\ A(adj\ A)$

$$= \frac{1}{|A|}\ |A|I = I$$

Similarly $BA = \frac{1}{|A|}\ (adj\ A)A = \frac{1}{|A|}\ A(adj\ A) = \frac{1}{|A|}\ |A|I = I$

Thus $AB = BA = I$ hence B is inverse of A and is given by $A^{-1} = \frac{1}{|A|}\ (adj\ A)$

Theorem 2.5.2: (Uniqueness of the Inverse)

Inverse of a matrix if it exists is unique.

Proof: Let B and C are inverse s of the matrix A then

$$AB = BA = I \text{ and } AC = CA = I$$

$$B(AC) = BI$$

$$(BA)C = B$$

$$C = B$$

Example 6: Let $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$ find A^{-1}

Theorem 2.5.3: (Reversal law of the inverse of product)

If A and B are two non-singular matrices of order n, then (AB) is also non singular and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof:

Since A and B are non-singular $|A| \neq 0$ and $|B| \neq 0$, therefore $|A||B| \neq 0$, then $|AB| \neq 0$.

Consider $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$$= AIA^{-1} = AA^{-1} = I \quad \text{...........(1)}$$

Similarly $(B^{-1}A^{-1})\ (AB) = B^{-1}(A^{-1}A)B$

$$= B^{-1}IB = B^{-1}B = I \quad \text{.............(2)}$$

From (1) and (2)

$$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})\ (AB) = I$$

Therefore by the definition and uniqueness of the inverse $(AB)^{-1} = B^{-1}A^{-1}$

Corollary 5.1: If $A_1A_2 \dots \dots \dots A_m$ are non singular matrices of order n, then $(A_1A_2 \dots \dots \dots A_m)^{-1} = A_1^{-1}A_2^{-2} \dots \dots \dots . A_m^{-1}$.

Theorem 2.5.4: If A is a non-singular matrix of order n then $(A^t)^{-1} = (A^{-1})^t$

Proof: Since $|A^t| = |A| \neq 0$ therefore the matrix A^t is non-singular and $(A^t)^{-1}$ exists.

Let $AA^{-1} = A^{-1}A = I$

Taking transpose on both sides we get

$$(AA^{-1})^t = (A^{-1})^tA^t = I^t{}_n = I_n$$

$$(A^{-1}A)^t = A^t(A^{-1})^t = I^t = I_n$$

Therefore $A^t(A^{-1})^t = (A^{-1})^tA^t = I_n$

That is $(A^{-1})^t = = (A^t)^{-1}$.

Theorem 2.5.5: If A is a non-singular matrix , k is non zero scalar, then $(kA)^{-1} = \frac{1}{k}A^{-1}$.

Proof: Since A is non-singular matrix A^{-1} exits.

Let consider $(kA) \left(\frac{1}{k}A^{-1}\right) = \left(k \times \frac{1}{k}\right) (A A^{-1}) = I$

Therefore $\left(\frac{1}{k}A^{-1}\right)$ is inverse of kA

By uniqueness if inverse $(kA)^{-1} = \frac{1}{k}A^{-1}$

Theorem 2.5.6: If A is a non-singular matrix then

$$|A^{-1}| = \frac{1}{|A|}.$$

Proof: Since A is non-singular matrix, A^{-1} exits and we have

$$AA^{-1} = I$$

Therefore $|AA^{-1}| = |A||A^{-1}| = |I| = 1$

Then $|A^{-1}| = \frac{1}{|A|}$

2.5.2 Elementary Transformations:

Some operations on matrices called as elementary transformations. There are six types of elementary transformations, three of then are row transformations and other three of them are column transformations. There are as follows

(i) Interchange of any two rows or columns.
(ii) Multiplication of the elements of any row (or column) by a non zero number k.
(iii) Multiplication to elements of any row or column by a scalar k and addition of it to the corresponding elements of any other row or column.

We adopt the following notations for above transformations

(i) Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$.
(ii) Multiplication by k to all elements in the i^{th} row $R_i \to kR_i$.
(iii) Multiplication to elements of j^{th} row by k and adding them to the corresponding elements of i^{th} row is denoted by $R_i \to R_i + kR_j$.

2.5.3 Equivalent Matrix:

A matrix B is said to be equivalent to a matrix A if B can be obtained from A, by for forming finitely many successive elementary transformations on a matrix A.

Denoted by A~ B.

2.5.4 Rank of a Matrix:

Definition 5.2:

A positive integer 'r' is said to be the rank of a non- zero matrix A if

(i) There exists at least one non-zero minor of order r of A and
(ii) Every minor of order greater than r of A is zero.

The rank of a matrix A is denoted by $\rho(A)$.

2.5.5 Echelon Matrices:

Definition 5.3:

A matrix $A = (a_{ij})$ is said to be echelon form (echelon matrix) if the number of zeros preceding the first non zero entry of a row increasing by row until zero rows remain.

In particular, an echelon matrix is called a row reduced echelon matrix if the distinguished elements are

(i) The only non- zero elements in their respective columns.
(ii) Each equal to 1.

Remark: The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.

Example 7:

Reduce following matrices to row reduce echelon form

(i) $A = \begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$

(ii) $B = \begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{pmatrix}$

2.6: Solution of System of Linear Equation by Matrix Method

2.6.1 Solution of the linear system *AX*= *B*

We now study how to find the solution of system of m linear equations in n unknowns.

Consider the system of equations in unknowns $x_1, x_2, \dots\dots\dots\dots\dots, x_n$ as

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots\dots . a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots\dots . a_{2n}x_n = b_2$$

…………………………………………………

$$a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots\dots . a_{nn}x_n = b_n$$

is called system of linear equations with n unknowns $x_1, x_2, \dots \dots \dots \dots \dots \dots, x_n$. If the constants $b_1, b_2 \dots \dots \dots \dots \dots \dots b_n$ are all zero then the system is said to be homogeneous type.

The above system can be put in the matrix form as

$$AX = B$$

Where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ X = $\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$ B= $\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}$

The matrix $A = (a_{ij})_{n\times n}$ is called coefficient matrix, the matrix X is called matrix of unknowns and B is called as matrix of constants, matrices X and B are of order $n \times 1$.

Definition 6.1: (**consistent**)

A set of values of $x_1, x_2, \dots \dots \dots \dots \dots \dots, x_n$ which satisfy all these equations simultaneously is called the solution of the system. If the system has at least one solution then the equations are said to be consistent otherwise they are said to be inconsistent.

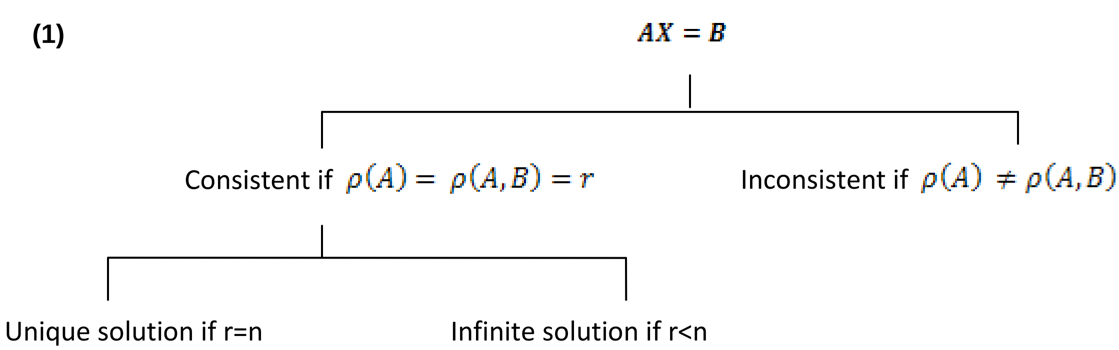
Theorem 2.5.1:

A system of m equations in n unknowns represented by the matrix equation AX= *B* is consistent if and only if $\rho(A) = \rho(A, B)$. That is the rank of matrix A is equal to rank of augment matrix (A, B)

Theorem 2.5.2:

If A be an non-singular matrix, X be an $n \times 1$ matrix and B be an $n \times 1$ *m*atrix then the system of equations *AX*= *B* has a unique solution.

(1)



(2)

$AX = 0$ (All wayes consistant)

If $\rho(A) = r = n$ Trivial solution

If $\rho(A) = r < n$ Infinite solutions

Therefore every system of linear equations solutions under one of the following:

(i) There is no solution
(ii) There is a unique solution
(iii) There are more than one solution

Methods of solving system of linear Equations:

6.1 Method of inversesion:

Consider the matrix equation

Consider the matrix equation

$AX = B$ Where $|A| \neq 0$

Pre multiplying by A^{-1}, we have

$$A^{-1}\ (AX\ ) = A^{-1}B$$

$$X = A^{-1}B$$

Thus $AX = B$, has only one solution if $|A| \neq 0$ and is given by $X = A^{-1}B$.

6.2 Using Elementary row operations: (Gaussian Elimination)

Suppose the coefficient matrix is of the type $m \times n$. That is we have m equations in n unknowns Write matrix $[A, B]$ and reduce it to Echelon augmented form by applying elementary row transformations only.

Example 8: Solve the following system of linear equations using matrix method

(i)

$$2x + y + -2z = 10$$
$$y + 10z = -28$$
$$3y + 16z = -42$$

(ii)

$$x + 2y - 3z = -1$$
$$3x - y + 2z = 7$$
$$5x + 3y - 4z = 2$$

Example 9: Determine the values of a so that the following system in unknowns x, y and z has

(i) No solutions
(ii) More than one solutions
(iii) A unique solution

$$x + y + z = 0$$

$$2x + 3y + az = 0$$

$$x + ay + 3z = 0$$

2.7: Eigen values and Eigenvectors:

If A is a square matrix of order n and X is a vector in R^n, (X considered as column matrix $n \times 1$), we are going to study the properties of non-zero X, where AX are scalar multiples of one another. Such vectors arise naturally in the study of vibrations, electrical systems, genetics, chemical reactions, quantum mechanics, economics and geometry.

Definition 7.1:
If A is a square matrix of order n , then a non-zero vector X in R^n is called eigenvector of A if $AX = \lambda X$ for some scalar λ. The scalar λ is called an eigenvalue of A, and X is said to be an eigenvector of A corresponding to λ.

Remark: Eigen values are also called proper values or characteristic values.

Example 10: The vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of A= $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

Theorem 2.7.1: If A is a square matrix of order n and λ is a real number, then λ is an eigenvalue of A if and only if $|\lambda I - A| = 0$

Proof: If λ is an eigenvalue of A, the there exist a non-zero X a vector in R^n such that $\boldsymbol{AX} = \lambda X$.

$$\boldsymbol{AX} = \lambda X$$

$\boldsymbol{AX} = \lambda I X$ Where I is a identity matrix of order n.

$$(\lambda I - A)X = 0$$

The equation has trivial solution when if and only if $|A| = 0$. The equation has non-zero solution if and only if $|(A - \lambda I)| = 0$.

Conversely , if $|(A - \lambda I)| = 0$ then by the result there will be a non-zero solution for the equation,

$$(A - \lambda I)X = 0$$

That is, there will a non-zero X in R^n such that $AX = \lambda X$, which shows that λ is an eigenvalue of A.

Example 11: Find the eigen values of the matrixes

(i) A = $\begin{pmatrix} 2 & 7 \\ 1 & -2 \end{pmatrix}$ (ii) B = $\begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{pmatrix}$

Theorem 2.7.2:

If A is an $n \times n$ matrix and λ is a real number, then the following are equivalent:

(i) λ is an eigenvalue of A.
(ii) The system of equations $(\lambda I - A)X = 0$ has non-trivial solutions.
(iii) There is a non-zero vector X in R^n such that $AX = \lambda I$.
(iv) Is a solution of the characteristic equation $|(A - \lambda I)|$ = 0.

Definition 7.2:

Let A be an the $n \times n$ matrix and λ be the eigen value of A. The set of all vectors X in R^n which satisfy identity $\boldsymbol{AX} = \lambda X$ is called the eigen space of a corresponding to λ. This is denoted by $E(\lambda)$.

Remark:

The eigenvectors of A corresponding to an eigen value λ are the non-zero vectors of X that satisfy $AX = \lambda I$. Equivalently the eigen vectors corresponding to λ are the non zero in the solution space of $(\lambda I - A)X = 0$. Therefore, the eigen space is the set of all non-zero X that satisfy $(A - \lambda I)X = 0$ with trivial solution in addition.

Steps to obtain eigen values and eigen vectors

Step I : For all real numbers λ form the matrix $\lambda I - A$

Step II: Evaluate $|(A - \lambda I)|$ That is characteristic polynomial of *A*.

Step III: Consider the equation $|(A - \lambda I)| = 0$ (The characteristic equation of *A*) Solve the equation for Let $\lambda_1, \lambda_2, \lambda_3 \dots \dots \dots \lambda_n$ be eigen values of A thus calculated.

Step IV: For each λ_i consider the equation $(\lambda_i I - A)X = 0$

Find the solution space of this system which an eigen space $E(\lambda_i)$ of A, corresponding to the eigen value λ_i of A . Repeat this for each λ_i $i = 1, 2, \dots \dots . n$

Step V: From step IV , we can find basis and dimension for each eigen space $E(\lambda_i)$ for $i = 1, 2, \dots \dots . n$

Example 12:

Find (i) Characteristic polynomial

(ii) Eigen values
(iii) Basis for the eigen space of a matrix

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

Example 13:

Find eigen values of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{pmatrix}$$

Also eigen space corresponding to each value of A. Further find basis and dimension for the same.

2.7.2 Diagonalization:

Definition 7.3: A square matrix *A* is called diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix, the matrix P is said to diagonalizable A.

Theorem 2.7.3: If A is a square matrix of order n, then the following are equivalent.

(i) A is diagonizible.
(ii) A has n linearly independent eigenvectors.

Procedure for diagonalizing a matrix

Step I: Find n linearly independent eigenvectors of A, say $P_1, P_2, \dots\dots\dots..P_n$

Step II: From the matrix P having $P_1, P_2, \dots\dots\dots..P_n$ as its column vectors.

Step III: The matrix $P^{-1}AP$ will then be diagonal with $\lambda_1, \lambda_2 \dots\dots..\lambda_n$ as its successive diagonal entries, where λ_i is the eigenvalue corresponding to P_i, $i = 1, 2, \dots\dots\dots..n$.

Example 14: Find a matrix P that diagonalizes

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{pmatrix}$$

Exercise

1. Show that the square matrix $A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 5 & 3 \\ -1 & 8 & 2 \end{pmatrix}$ is a singular matrix.

2. If $A = \begin{pmatrix} 1 & 4 & 3 \\ 6 & 2 & 5 \\ 1 & 7 & 0 \end{pmatrix}$ determine (i) $|A|$ (ii) *Adj A*

3. Find the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 5 & 1 \\ 2 & 0 & 6 \end{pmatrix}$

4. If $A = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 0.1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ determine

   (i) B^{-1} (ii) AB (iii) $B^{-1}A$

5. Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{pmatrix}$

   (i) Compute $|A|$ (ii) find adj A

   (iii) Verify $A(adj\ A) = |A|$ I (iv) Find A^{-1}

6. Find the possible value of x can take, given that

   $A = \begin{pmatrix} x^2 & 3 \\ 1 & 3x \end{pmatrix}$ $B = \begin{pmatrix} 3 & 6 \\ 2 & x \end{pmatrix}$ such that $AB = BA$.

7. If $A = \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$ find the values of m and n given that $A^2 = mA + nA$

8. Find the echelon form of matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 9 & 16 & 25 \end{pmatrix}$$

Hence discuss (1) unique solution (ii) many solutions and (iii) No solutions of the following system and solve completely.

$x + y + z = 1$

$2x + 3y + 4z = 5$

$4x + 9y + 16z = 25$

9. If matrix A is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{pmatrix}$ and I, the unit matrix of order 3, show that $A^3 = pI + qA + rA^2$.

10. Let A be a square matrix

a. Show that
$$(I - A)^{-1} = 1 + A + A^2 + A^3 \quad if \quad A^4 = 0$$

b. Show that
$$(I - A)^{-1} = 1 + A + A^2 + A^3 + \cdots \ldots . A^n \quad if \quad A^{n+1} = 0$$

11. Find values of a,b and c so that the graph of the polynomial $p(x) = ax^2 + bx + c$ passes through the points (1,2), (-1,6) and (2,3).

12. Find values of a,b and c so that the graph of the polynomial $p(x) = ax^2 + bx + c$passes through the points (-1,0) and has a horizontal tangent at (2,-9).

13. Let $\begin{pmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{pmatrix}$ be the augmented matrix for a linear- system. For what value of a and b does the system have

a. a unique solution
b. a one- parameter solution
c. a two parameter- solution
d. no solution

14. Find a matrix K such that $AKB = C$ given that

$$A = \begin{pmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 8 & 6 & -6 \\ 6 & -1 & 1 \\ -4 & 0 & 0 \end{pmatrix}$$

a. For the triangle below, use trigonometry to show
$$b \cos\gamma \; + \; c \cos\beta \; = a$$
$$c \cos\alpha \; + \; a \cos\gamma \; = b$$
$$a \cos\beta \; + \; b \cos\alpha \; = c$$

And then apply Crame's Rule to show

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

Use the Cramer's rule to obtain similar formulas for $\cos\beta$ and $\cos\gamma$.

Chapter 3

Ordinary Differential Equations

3.1 First Order Ordinary Differential Equations

Relationship between rate of change of variables rather than variables themselves gives rise to differential equations. Mathematical formulation of most of the physical and engineering problems leads to differential equations. It is very important for engineers and scientists to know inception and solving of differential equations. These are of two types:

1) Ordinary Differential Equations (ODE)
2) Partial Differential Equations (PDE)

An ordinary differential equation (ODE) involves the derivatives of a dependent variable w.r.t. a single independent variable whereas a partial differential equation (PDE) contains the derivatives of a dependent variable w.r.t. two or more independent variables. In this chapter we will confine our studies to ordinary differential equations.

Prelims:

- $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$
- $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$
- $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$
- $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$
- $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$
- If u and v are functions of x and u vanishes after a finite number of differentiations

$$\int u.v\,dx = uv_1 - u^{(1)}v_2 + u^{(2)}v_3 - u^{(3)}v_4 + \cdots$$

Here $u^{(n)}$ is derivative of $u^{(n-1)}$ and v_n is integral of v_{n-1}

For example

$$\int x^2.\sin x\,dx = (x^2)(-\cos x) - (2x)(-\sin x) + (2)(\cos x)$$

$$= -x^2\cos x + 2x\sin x + 2\cos x$$

Order and Degree of Ordinary Differential Equations (ODE)

A general ODE of n^{th} order can be represented in the form $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right)=0$ Order of an ordinary differential equation is that of the highest derivative occurring in it and the degree is the power of highest derivative after it has been freed from all radical signs.

The differential equation $\left(\frac{d^2y}{dx^2} + 2y\right)^3 + \frac{d^3y}{dx^3} + y = 0$ is having order 3 and degree 1.

Whereas $\left(\frac{d^3y}{dx^3} + 2y\right)^3 + \frac{d^2y}{dx^2} + y = 0$ is having order 3 and degree 3.

The differential equation $\sqrt{\frac{d^2y}{dx^2}} = \frac{d^3y}{dx^3} + y$ is of order 3 and degree 2.

3.2 First Order Linear Differential Equations (Leibnitz's Linear Equations)

A first order linear differential equation is of the form $\frac{dy}{dx} + Py = Q$, ...Ⓐ

where P and Q are functions of x alone or constants. To solve Ⓐ, multiplying throughout by $e^{\int P\,dx}$ (here $e^{\int P\,dx}$ is known as Integrating Factor (IF)), we get

$$\frac{dy}{dx}e^{\int P\,dx} + Py\,e^{\int P\,dx} = Q\,e^{\int P\,dx}$$

$$\Rightarrow \mathrm{d}\left(y\,e^{\int P\,dx}\right) = Q\,e^{\int P\,dx}$$

$$\Rightarrow y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + C$$

Algorithm to solve a first order linear differential equation (Leibnitz's Equation)

1. Write the given equation in standard form i.e. $\frac{dy}{dx} + Py = Q$
2. Find the integrating factor (IF) = $e^{\int P\,dx}$
3. Solution is given by y. IF = $\int Q$ .IF dx + C , C is an arbitrary constant

Note: If the given equation is of the type $\frac{dx}{dy} + Px = Q$,

then IF = $e^{\int P\,dy}$ and the solution is given by x. IF = $\int Q$ .IF dy + C

Example 1 Solve the differential equation: $\frac{dy}{dx} = \frac{x + y\sin x}{1+\cos x}$

Solution: The given equation may be written as:

$\frac{dy}{dx} - \frac{\sin x}{1+\cos x} y = \frac{x}{1+\cos x}$ ...... ①

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = -\frac{\sin x}{1+\cos x}$ and $Q = \frac{x}{1+\cos x}$

IF = $e^{\int P\,dx} = e^{\int -\frac{\sin x}{1+\cos x} dx} = e^{\log|1+\cos x|} = 1 + \cos x$

∴ Solution of ① is given by

$y.(1+\cos x) = \int \frac{x}{1+\cos x}(1+\cos x)\,dx + C$

$\Rightarrow y\,(1+\cos x) = \frac{x^2}{2} + C$

Example 2 Solve the differential equation: $\frac{dy}{dx} = (1+x) + (1-y)$

Solution: The given equation may be written as:

$\frac{dy}{dx} + y = 2 + x$ .... ①

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = 1$ and $Q = 2 + x$

IF = $e^{\int P\,dx} = e^{\int dx} = e^x$

∴ Solution of ① is given by

$y.\,e^x = \int (2+x)e^x\,dx + C$

$\Rightarrow y = 1 + x + Ce^{-x}$

Example 3 Solve the differential equation: $(x+y+1)\frac{dy}{dx} = 1$

Solution: The given equation may be written as:

$\frac{dx}{dy} = x + y + 1 \quad \Rightarrow \quad \frac{dx}{dy} - x = y + 1$ ......①

This is a linear differential equation of the form $\frac{dx}{dy} + Px = Q$

Where $P = -1$ and $Q = y + 1$

IF = $e^{\int P\,dy} = e^{\int -dy} = e^{-y}$

∴ Solution of ① is given by

$x.\,e^{-y} = \int (y+1)e^{-y}\,dy + C$

$\Rightarrow x\,e^{-y} = -(y+2)e^{-y} + C$

$\Rightarrow x = -(y+2) + C\,e^{y}$

Example 4 Solve the differential equation: $x\log x \frac{dy}{dx} + y = 2\log x$

Solution: The given equation may be written as:

$\frac{dy}{dx} + \frac{1}{x\log x} y = \frac{2}{x}$ ....①

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = \frac{1}{x\log x}$ and $Q = \frac{2}{x}$

IF = $e^{\int P\,dx} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$

∴ Solution of ① is given by

$y.\log x = \int \frac{2}{x}\log x\ dx + C$

$\Rightarrow y\log x = (\log x)^2 + C$, C is an arbitrary constant

Example 5 Solve the differential equation: $\frac{dy}{dx} = \frac{e^{2\sqrt{x}}+y}{\sqrt{x}}$

Solution: The given equation may be written as:

$\frac{dy}{dx} - \frac{1}{\sqrt{x}} y = \frac{e^{2\sqrt{x}}}{\sqrt{x}}$ ......①

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = -\frac{1}{\sqrt{x}}$ and $Q = \frac{e^{2\sqrt{x}}}{\sqrt{x}}$

IF = $e^{\int P\,dx} = e^{\int \frac{-1}{\sqrt{x}}dx} = e^{-2\sqrt{x}}$

∴ Solution of ① is given by

$y.\, e^{-2\sqrt{x}} = \int \frac{e^{2\sqrt{x}}}{\sqrt{x}} e^{-2\sqrt{x}}\, dx + C$

$\Rightarrow y.\, e^{-2\sqrt{x}} = \int \frac{1}{\sqrt{x}}\, dx + C$

$\Rightarrow y.\, e^{-2\sqrt{x}} = 2\sqrt{x} + C$

$\Rightarrow y = 2\sqrt{x}\ e^{2\sqrt{x}} + C\ e^{2\sqrt{x}}$

3.3 Equations Reducible to Leibnitz's Equations (Bernoulli's Equations)

Differential equation of the form $\frac{dy}{dx} + Pf(y) = Qg(y)$, ...Ⓑ

where P and Q are functions of x alone or constant, is called Bernoulli's equation. Dividing both sides of Ⓑ by $g(y)$, we get $\frac{1}{g(y)}\frac{dy}{dx} + P\frac{f(y)}{g(y)} = Q$. Now putting $\frac{f(y)}{g(y)} = t$, Ⓑ reduces to Leibnitz's equation.

Example 6 Solve the differential equation: $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$ ......①

Solution: The given equation may be written as:

$e^{-y}\frac{dy}{dx} + \frac{1}{x}e^{-y} = \frac{1}{x^2}$ ......②

Putting $e^{-y} = t$, $-e^{-y}\frac{dy}{dx} = \frac{dt}{dx}$ ......③

Using ③ in ②, we get $\frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x^2}$ ....④

④ is a linear differential equation of the form $\frac{dt}{dx} + Pt = Q$

Where $P = -\frac{1}{x}$ and $Q = -\frac{1}{x^2}$

$\text{IF} = e^{\int P\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

∴ Solution of ④ is given by

$t.\frac{1}{x} = \int -\frac{1}{x^2}.\frac{1}{x}\,dx + C$

$\Rightarrow t.\frac{1}{x} = \frac{1}{2x^2} + C$

Substituting $t = e^{-y}$

$\Rightarrow \frac{e^{-y}}{x} = \frac{1}{2x^2} + C$

$\Rightarrow 2x = e^y(2cx^2 + 1)$

Example 7 Solve the differential equation:

$\tan y\frac{dy}{dx} + \tan x = \cos y cos^3 x$ ......①

Solution: The given equation may be written as:

$\frac{\tan y}{\cos y}\frac{dy}{dx} + \frac{\tan x}{\cos y} = cos^3 x$

$\Rightarrow \sec y\ \tan y\frac{dy}{dx} + \sec y\ \tan x = cos^3 x$ ......②

Putting $\sec y = t$, $\sec y\ \tan y\frac{dy}{dx} = \frac{dt}{dx}$ ......③

Using ③ in ②, we get $\frac{dt}{dx} + (\tan x)\,t = cos^3 x$ ....④

④ is a linear differential equation of the form $\frac{dt}{dx} + Pt = Q$

Where $P = \tan x$ and $Q = cos^3 x$

$\text{IF} = e^{\int P\,dx} = e^{\int \tan x\,dx} = e^{\log|\sec x|} = \sec x$

∴ Solution of ④ is given by

$t.\sec x = \int cos^3 x.\sec x \;\; dx + C$

$\Rightarrow t.\sec x = \int cos^2 x \;\; dx + C$

$\Rightarrow t.\sec x = \int \frac{1+\cos 2x}{2} \;\; dx + C$

$\Rightarrow t.\sec x = \frac{x}{2} + \frac{\sin 2x}{4} + C$

Substituting $t = \sec y$,

$\Rightarrow \sec x \sec y = \frac{x}{2} + \frac{\sin 2x}{4} + C$

Example 8 Solve the differential equation: $\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$ ......①

Solution: The given equation may be written as:

$\frac{dx}{dy} = \frac{x+\sqrt{xy}}{y}$

$\Rightarrow \frac{dx}{dy} - \frac{1}{y}x = \sqrt{\frac{x}{y}}$

Dividing throughout by $\sqrt{x}$

$\Rightarrow \frac{1}{\sqrt{x}}\frac{dx}{dy} - \frac{1}{y}\sqrt{x} = \frac{1}{\sqrt{y}}$ ......②

Putting $\sqrt{x} = t$, $\frac{1}{2\sqrt{x}}\frac{dx}{dy} = \frac{dt}{dy}$ ......③

Using ③ in ②, we get $\frac{dt}{dy} - \frac{1}{2y}t = \frac{1}{2\sqrt{y}}$ ....④

④ is a linear differential equation of the form $\frac{dt}{dy} + Pt = Q$

Where $P = -\frac{1}{2y}$ and $Q = \frac{1}{2\sqrt{y}}$

$\text{IF} = e^{\int P\,dx} = e^{\int \frac{-1}{2y}dy} = e^{\frac{-1}{2}\log y} = e^{\log\frac{1}{\sqrt{y}}} = \frac{1}{\sqrt{y}}$

∴ Solution of ④ is given by

$t.\frac{1}{\sqrt{y}} = \int \frac{1}{2\sqrt{y}}.\frac{1}{\sqrt{y}} \;\; dx + C$

$\Rightarrow t.\frac{1}{\sqrt{y}} = \int \frac{1}{2y} \;\; dx + C$

$\Rightarrow t.\frac{1}{\sqrt{y}} = \frac{1}{2}\, log\, y + C$

Substituting $t = \sqrt{x}$

$$\sqrt{\frac{x}{y}} = log\sqrt{y} + C$$

Example 9 Solve the differential equation: $x\frac{dy}{dx} + y = y^2 \log x$ .....①

Solution: The given equation may be written as:

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x}\log x$$

Dividing throughout by y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x}\log x \text{......②}$$

Putting $\frac{1}{y} = t,\ -\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$ ......③

Using ③ in ②, we get $\frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x}\log x$ ....④

④ is a linear differential equation of the form $\frac{dt}{dx} + Pt = Q$

Where $P = -\frac{1}{x}$ and $Q = -\frac{1}{x}\log x$

$$\text{IF} = e^{\int P\,dx} = e^{\int \frac{-1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$$

∴ Solution of ④ is given by

$$t.\frac{1}{x} = \int -\frac{1}{x}\log x.\frac{1}{x}\ dx + C$$

$$\Rightarrow t.\frac{1}{x} = \int -\frac{1}{x^2}\log x\ dx + C$$

Putting $\log x = u, \frac{1}{x}dx = du$, also $x = e^u$

$$\Rightarrow t.\frac{1}{x} = -\int ue^{-u}\ du + C$$

$$\Rightarrow t.\frac{1}{x} = -[u(-e^{-u}) - 1(e^{-u})] + C$$

$$\Rightarrow t.\frac{1}{x} = e^{-u}(u+1) + C$$

$$\Rightarrow t.\frac{1}{x} = \frac{1}{x}(log\,x + 1) + C$$

Substituting $t = \frac{1}{y}$

$$\Rightarrow \frac{1}{xy} = \frac{1}{x}(\log x + 1) + C$$

$\Rightarrow \frac{1}{y} = (\log x + 1) + C\,x$, C is an arbitrary constant

Exercise 3.1

Solve the following differential equations:

1. $e^{-y} sec^2 y\, dy = dx + xdy$

   Ans. $\langle xe^y = C + \tan y \rangle$

2. $(x+1)\frac{dy}{dx} - 2y = (x+1)^4$

   Ans. $\langle y = \left(\frac{x^2}{2} + x + c\right)(x+1)^2 \rangle$

3. $\frac{dy}{dx} = \frac{e^{-2\sqrt{x}} - y}{\sqrt{x}}$ Ans. $y.\, e^{2\sqrt{x}} = 2\sqrt{x} + C$
4. $\frac{dx}{dy} = \left(\frac{\sqrt{1+y^2}\sin y - xy}{1+y^2}\right)$ Ans. $x\sqrt{1+y^2} + \cos y = C$
5. $(x+2y^3)\frac{dy}{dx} = y$ Ans. $x = y^3 + Cy$
6. $\frac{dy}{dx}\cos x + y\sin x = \sqrt{y\sec x}$ Ans. $2\sqrt{y\sec x} = \tan x + 2C$
7. $\frac{dy}{dx} - xy + y^3 e^{-x^2} = 0$ Ans. $e^{x^2} = y^2(2x - C)$
8. $3x(1-x^2)y^2\frac{dy}{dx} + (2x^2-1)y^3 = x^3$

   Ans. $y^3 = x + Cx\sqrt{1-x^2}$

9. $\frac{dy}{dx} + y\cos x = y^n \sin 2x$ Ans. $(y^{-n+1} - 1)e^{\sin x} = C$
10. $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$ Ans. $e^y = Ce^{-e^x} + e^x - 1$

3.4 Exact Differential Equations of First Order

A differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if it can be directly obtained from its primitive by differentiation.

Theorem: The necessary and sufficient condition for the equation $M(x,y)dx + N(x,y)dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Working rule to solve an exact differential equation

1. For the equation $M(x,y)dx + N(x,y)dy = 0$, check the condition for exactness i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. Solution of the given differential equation is given by

$$\int M(taking\ y\ as\ constant)\, dx + \int N(terms\ not\ containg\ x)dy = C$$

Example 10 Solve the differential equation:

$(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$ .....①

Solution: $M = (e^y + 1)\cos x$, $N = e^y \sin x$

$\frac{\partial M}{\partial y} = e^y \cos x$, $\frac{\partial N}{\partial x} = e^y \cos x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$$\int_{y\ constant} (e^y + 1)\cos x \ dx + \int 0 \, dy = C$$

$\Rightarrow (e^y + 1)\sin x = C$

Example 11 Solve the differential equation:

$(\sec x \tan x \ \tan y - e^x)dx + (\sec x \ sec^2 y)dy = 0$ .....①

Solution: $M = \sec x \tan x \ \tan y - e^x$, $N = \sec x \ sec^2 y$

$\frac{\partial M}{\partial y} = \sec x \tan x \, sec^2 y$, $\frac{\partial N}{\partial x} = \sec x \tan x \, sec^2 y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$$\int_{y\ constant} (\sec x \tan x \ \tan y - e^x)\, dx + \int 0 \, dy = C$$

$\Rightarrow \sec x \tan y - e^x = C$

Example 12 Solve the differential equation:

$\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + [x + \log x - x \sin y]dy = 0$ .....①

Solution: $M = y\left(1 + \frac{1}{x}\right) + \cos y$, $N = x + \log x - x \sin y$

$\frac{\partial M}{\partial y} = \left(1 + \frac{1}{x}\right) - \sin y$, $\frac{\partial N}{\partial x} = \left(1 + \frac{1}{x}\right) - \sin y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$$\int_{y\ constant} \left[y\left(1 + \frac{1}{x}\right) + \cos y\right] dx + \int 0 \, dy = C$$

$\Rightarrow y\,(x + \log x) + x \cos y = C$

Example 13 Solve the differential equation:

$x\,dx + y\,dy = \frac{a^2(x\,dy - y\,dx)}{x^2+y^2}$ .....①

Solution: $\Rightarrow \left(x + \frac{a^2 y}{x^2+y^2}\right)dx + \left(y - \frac{a^2 x}{x^2+y^2}\right)dy = 0$

$M = x + \frac{a^2 y}{x^2+y^2}$, $N = y - \frac{a^2 x}{x^2+y^2}$

$\frac{\partial M}{\partial y} = \frac{a^2(x^2-y^2)}{(x^2+y^2)^2}$, $\frac{\partial N}{\partial x} = \frac{a^2(x^2-y^2)}{(x^2+y^2)^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$$\int_{y\ constant} \left(x + \frac{a^2 y}{x^2+y^2}\right) dx + \int y\, dy = C$$

$\Rightarrow \frac{x^2}{2} + a^2 \tan^{-1}\frac{x}{y} + \frac{y^2}{2} = C$

$\Rightarrow x^2 + 2a^2 \tan^{-1}\frac{x}{y} + y^2 = D$, $D = 2C$

3.5 Equations Reducible to Exact Differential Equations

Sometimes a differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is not exact i.e. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. It can be made exact by multiplying the equation by some function of x and y known as integrating factor (IF).

3.5.1 Integrating Factor (IF) Found By Inspection

Some non-exact differential equations can be grouped or rearranged and solved directly by integration, after multiplying by an integrating factor (IF) which can be found just by inspection as shown below:

Term	IF	Result
$xdy + ydx$	1. $\frac{1}{xy}$	$\frac{xdy + ydx}{xy} = \frac{1}{y}dy + \frac{1}{x}dx = d[\log(xy)]$
	2. $\frac{1}{(xy)^n}, n \neq 1$	$\frac{xdy + ydx}{(xy)^n} = \frac{d(xy)}{(xy)^n} = -d\left[\frac{1}{(n-1)(xy)^{n-1}}\right]$
$xdy - ydx$	1. $\frac{1}{x^2}$	$\frac{xdy - ydx}{x^2} = d\left[\frac{y}{x}\right]$
	2. $\frac{1}{y^2}$	$\frac{xdy - ydx}{y^2} = -d\left[\frac{x}{y}\right]$
	3. $\frac{1}{xy}$	$\frac{xdy - ydx}{xy} = d\left[\log\frac{y}{x}\right]$
	4. $\frac{1}{x^2+y^2}$	

	5. $\frac{1}{x\sqrt{x^2-y^2}}$	$\frac{xdy-ydx}{x^2+y^2}=d\left[tan^{-1}\frac{y}{x}\right]$ $\frac{xdy-ydx}{x\sqrt{x^2-y^2}}=d\left[sin^{-1}\frac{y}{x}\right]$
$xdx+ydy$	1. $\frac{1}{x^2+y^2}$ 2. $\frac{1}{(x^2+y^2)^n}, n\neq 1$	$\frac{xdx+ydy}{x^2+y^2}=\frac{1}{2}d[log(x^2+y^2)]$ $\frac{xdx+ydy}{(x^2+y^2)^n}=\frac{1}{2}d\left[\frac{(x^2+y^2)^{-n+1}}{-n+1}\right]$

Example 14 Solve the differential equation:

$x\,dy-\,y\,dx\,+2x^3dx\,=0$ .....①

Solution: $\Rightarrow(-y+2x^3)dx+\,xdy=0$

$M=\,-y+\,2x^3\,,\;N=\,x$

$\frac{\partial M}{\partial y}=\,-1\,,\;\frac{\partial N}{\partial x}=\,1$

$\frac{\partial M}{\partial y}\neq\,\frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Taking $\frac{1}{x^2}$ as integrating factor due to presence of the term $(x\,dy-\,y\,dx)$

① may be rewritten as : $\frac{xdy-ydx}{x^2}+\,2x\,dx=0$

$\Rightarrow\,d\left[\frac{y}{x}\right]+\,2x\,dx=0$ ......②

Integrating ②, solution is given by : $\frac{y}{x}+x^2=C$

$\Rightarrow y+x^3=Cx$

Example 15 Solve the differential equation:

$y\,dx-\,x\,dy\,+(1+x^2)dx\,+\,x^2\cos y\,dy\,=0$ .....①

Solution: $\Rightarrow(y+1+\,x^2)dx+\,(x^2\cos y\,-x)dy=0$

$M=\,y+1+\,x^2,\;N=\,x^2\cos y\,-x$

$\frac{\partial M}{\partial y}=\,1\,,\;\frac{\partial N}{\partial x}=\,2x\cos y\,-1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Taking $\frac{1}{x^2}$ as integrating factor due to presence of the term $(y\,dx - x\,dy)$

① may be rewritten as : $\frac{y dy - x dx}{x^2} + \left(\frac{1}{x^2} + 1\right) dx + \cos y\, dy = 0$

$\Rightarrow -d\left[\frac{y}{x}\right] + \left(\frac{1}{x^2} + 1\right) dx + \cos y\, dy = 0$ ......②

Integrating ②, solution is given by : $-\frac{y}{x} + \left(-\frac{1}{x} + x\right) + \sin y = C$

$\Rightarrow x^2 - y - 1 + x\sin y = Cx$

Example 16 Solve the differential equation:

$x\,dx + y\,dy = a(x^2 + y^2)dy$ .....①

Solution: $\Rightarrow xdx + (y - a(x^2 + y^2))dy = 0$

$M = x$, $N = y - a(x^2 + y^2)$

$\frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = -2ax$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Taking $\frac{1}{x^2 + y^2}$ as integrating factor due to presence of the term $(x\,dx + y\,dy)$

① may be rewritten as : $\frac{xdx + ydy}{x^2 + y^2} - a\,dy = 0$

$\Rightarrow \frac{1}{2} d[log(x^2 + y^2)] - a\,dy = 0$

$\Rightarrow d[log(x^2 + y^2)] - 2\,a\,dy = 0$ ......②

Integrating ②, solution is given by: $(x^2 + y^2) - 2ay = C$, C is an arbitrary constant

Example 17 Solve the differential equation:

$a(x\,dy + 2y\,dx) = xy\,dy$ .....①

Solution: $\Rightarrow 2aydx + (ax - xy)dy = 0$

$M = 2ay$, $N = ax - xy$

$\frac{\partial M}{\partial y} = 2a$, $\frac{\partial N}{\partial x} = a - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rewriting ① as $a(x\,dy + y\,dx) + ay\,dx = xy\,dy$ ......②

Taking $\frac{1}{xy}$ as integrating factor due to presence of the term $(x\,dy + y\,dx)$

② may be rewritten as : $a\frac{xdy+ydx}{xy} + \frac{a}{x}dx - dy = 0$

$\Rightarrow ad[\log(xy)] + \frac{a}{x}dx - dy = 0$ ......③

Integrating ③ solution is given by: $a\log(xy) + a\log x - y = C$

$\Rightarrow a\log(x^2y) - y = C$, C is an arbitrary constant

Example 18 Solve the differential equation:

$$x^4\frac{dy}{dx} + x^3y + \sec(xy) = 0 \quad .....①$$

Solution: $\Rightarrow (x^3y + \sec(xy))dx + x^4dy = 0$

$M = x^3y + \sec(xy)$, $N = x^4$

$\frac{\partial M}{\partial y} = x^3 + x\sec(xy)\tan(xy)$, $\frac{\partial N}{\partial x} = 4x^3$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rewriting ① as: $x^3(x\,dy + y\,dx) + \sec(xy)\,dx = 0$

$\Rightarrow \frac{(x\,dy+y\,dx)}{\sec(xy)} - x^{-3}dx = 0$

$\Rightarrow \cos(xy)(x\,dy + y\,dx) - x^{-3}dx = 0$

$\Rightarrow d[\sin(xy)] - \frac{1}{2}d(x^{-2})dx = 0$ ......②

Integrating ②, we get the required solution as:

$\sin(xy) - \frac{x^{-2}}{2} = C$

$\Rightarrow 2x^2\sin(xy) - 1 = Cx^2$

3.5.2 Integrating Factor (IF) of a Non-Exact Homogeneous Equation

If the equation $Mdx + Ndy = 0$ is a homogeneous equation, then the integrating factor (IF) will be $\frac{1}{Mx+Ny}$, provided $Mx + Ny \neq 0$

Example 19 Solve the differential equation:

$(x^3 + y^3)dx - xy^2\,dy = 0$ .....①

Solution: $M = x^3 + y^3$, $N = -xy^2$

$\frac{\partial M}{\partial y} = 3y^2$, $\frac{\partial N}{\partial x} = -y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is a homogeneous equation , ∴ IF $= \frac{1}{Mx+Ny} = \frac{1}{x^4+xy^3-xy^3} = \frac{1}{x^4}$

∴ ① may be rewritten as : $\left(\frac{1}{x}+\frac{y^3}{x^4}\right)dx - \frac{y^2}{x^3}dy = 0$ ......②

New $M = \frac{1}{x}+\frac{y^3}{x^4}$, New $N = -\frac{y^2}{x^3}$

$\frac{\partial M}{\partial y} = \frac{3y^2}{x^4}$, $\frac{\partial N}{\partial x} = \frac{3y^2}{x^4}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant}\left(\frac{1}{x}+\frac{y^3}{x^4}\right)dx + \int 0\,dy$$

$\Rightarrow \log x - \frac{1}{3}\left(\frac{y}{x}\right)^3 = C$

Example 20 Solve the differential equation:

$(3y^4 + 3x^2y^2)dx + (x^3y - 3xy^3)\,dy = 0$ .....①

Solution: $M = 3y^4 + 3x^2y^2$, $N = x^3y - 3xy^3$

$\frac{\partial M}{\partial y} = 12y^3 + 6x^2y$, $\frac{\partial N}{\partial x} = 3x^2y - 3y^3$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is a homogeneous equation

∴ IF $= \frac{1}{Mx+Ny} = \frac{1}{3xy^4+3x^3y^2+x^3y^2-3xy^4} = \frac{1}{4x^3y^2}$

∴ ① may be rewritten after multiplying by IF as:

$\left(\frac{3y^2}{4x^3}+\frac{3}{4x}\right)dx + \left(\frac{1}{4y}-\frac{3y}{4x^2}\right)dy = 0$ ......②

New $M = \frac{3y^2}{4x^3}+\frac{3}{4x}$, New $N = \frac{1}{4y}-\frac{3y}{4x^2}$

$\frac{\partial M}{\partial y} = \frac{6y}{4x^3} = \frac{3y}{2x^3}$, $\frac{\partial N}{\partial x} = \frac{3y}{2x^3}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant}\left(\frac{3y^2}{4x^3}+\frac{3}{4x}\right)dx + \int \frac{1}{4y}\,dy$$

$\Rightarrow \frac{-3y^2}{8x^2} + \frac{3}{4}\log x + \frac{1}{4}\log y = C$

$\Rightarrow \log x^3 y - \frac{3y^2}{2x^2} = D$, $D = 4C$

3.5.3 Integrating Factor of a Non-Exact Differential Equation of the Form

$\boldsymbol{y f_1(xy) dx + x f_2(xy)\, dy = 0}$ **:** If the equation $Mdx + Ndy = 0$ is of the given form, then the integrating factor (IF) will be $\frac{1}{Mx-Ny}$ provided $Mx - Ny \neq 0$

Example 21 Solve the differential equation:

$y(1 + xy)dx + x(1 - xy)\, dy = 0$ .....①

Solution: $M = y + xy^2$, $N = x - x^2 y$

$\frac{\partial M}{\partial y} = 1 + 2xy$, $\frac{\partial N}{\partial x} = 1 - 2xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

$\therefore \text{IF} = \frac{1}{Mx-Ny} = \frac{1}{xy+x^2y^2-xy+x^2y^2} = \frac{1}{2x^2y^2}$

∴ ① may be rewritten after multiplying by IF as:

$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0$ ......②

New $M = \frac{1}{2x^2y} + \frac{1}{2x}$, New $N = \frac{1}{2xy^2} - \frac{1}{2y}$

$\frac{\partial M}{\partial y} = \frac{-1}{2x^2y^2}$, $\frac{\partial N}{\partial x} = \frac{-1}{2x^2y^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant} \left(\frac{1}{2x^2y} + \frac{1}{2x}\right) dx + \int -\frac{1}{2y}\, dy$$

$\Rightarrow \frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = C$

$\Rightarrow \log\frac{x}{y} - \frac{1}{xy} = D$, $D = 2C$

Example 22 Solve the differential equation:

$y(xy + 2x^2y^2)dx + x(xy - x^2y^2)\, dy = 0$ .....①

Solution: $M = xy^2 + 2x^2y^3$, $N = x^2y - x^3y^2$

$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2$, $\frac{\partial N}{\partial x} = 2xy - 3x^2y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

$\therefore \text{IF} = \frac{1}{Mx-Ny} = \frac{1}{x^2y^2+2x^3y^3-x^2y^2+x^3y^3} = \frac{1}{3x^3y^3}$

∴ ① may be rewritten after multiplying by IF as:

$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$ ......②

New $M = \frac{1}{x^2y} + \frac{2}{x}$, New $N = \frac{1}{xy^2} - \frac{1}{y}$

$\frac{\partial M}{\partial y} = \frac{-1}{x^2y^2}$, $\frac{\partial N}{\partial x} = \frac{-1}{x^2y^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,∴ ② is an exact differential equation.

Solution of ② is given by:

$$\underset{y\ constant}{\int} \left(\frac{1}{x^2y} + \frac{2}{x}\right) dx + \int -\frac{1}{y}dy$$

$\Rightarrow \frac{-1}{xy} + 2\log x - \log y = C$

$\Rightarrow \log\frac{x^2}{y} - \frac{1}{xy} = C$

3.5.4 Integrating Factor (IF) of a Non-Exact Differential Equation

$Mdx + Ndx = 0$ in which $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are connected in a specific way as shown:

i. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, a function of x alone, then IF $= e^{\int f(x)dx}$

ii. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$, a function of y alone, then IF $= e^{\int -g(y)dy}$

Example 23 Solve the differential equation:

$(x^3 + y^2 + x)dx + xy\,dy = 0$ .....①

Solution: $M = x^3 + y^2 + x$, $N = xy$

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

$\therefore$ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{y}{xy} = \frac{1}{x} = f(x)$ say

$\therefore$ IF $= e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

$\therefore$ ① may be rewritten after multiplying by IF as:

$(x^4 + xy^2 + x^2)dx + x^2y\,dy = 0$ ....②

New $M = x^4 + xy^2 + x^2$, New $N = x^2y$

$\frac{\partial M}{\partial y} = 2xy$, $\frac{\partial N}{\partial x} = 2xy$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\therefore$ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant} (x^4 + xy^2 + x^2)\,dx + \int 0\,dy$$

$\Rightarrow \frac{x^5}{5} + \frac{x^2y^2}{2} + \frac{x^3}{3} = C$

Example 24 Solve the differential equation:

$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)\,dy = 0$ .....①

Solution: $M = y^4 + 2y$, $N = xy^3 + 2y^4 - 4x$

$\frac{\partial M}{\partial y} = 4y^3 + 2$, $\frac{\partial N}{\partial x} = y^3 - 4$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, $\therefore$ given differential equation is not exact.

As ① is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

$\therefore$ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3y^3 + 6$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{3y^3+6}{y^4+2y} = \frac{3}{y} = g(y)$ say

$\therefore$ IF $= e^{\int -g(y)dy} = e^{\int -\frac{3}{y}dy} = e^{-3\log y} = \frac{1}{y^3}$

$\therefore$ ① may be rewritten after multiplying by IF as:

$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$ ......②

New $M = y + \frac{2}{y^2}$, New $N = x + 2y - \frac{4x}{y^3}$

$\frac{\partial M}{\partial y} = 1 - \frac{4}{y^3}$, $\frac{\partial N}{\partial x} = 1 - \frac{4}{y^3}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,∴ ② is an exact differential equation.

Solution of ② is given by:

$$\underset{y\ constant}{\int}\left(y + \frac{2}{y^2}\right) dx + \int 2y\, dy$$

$$\Rightarrow \left(y + \frac{2}{y^2}\right)x + y^2 = C$$

Example 25 Solve the differential equation:

$(x^2 - y^2 + 2x)dx - 2y\, dy = 0$ .....①

Solution: $M = x^2 - y^2 + 2x$, $N = -2y$

$\frac{\partial M}{\partial y} = -2y$, $\frac{\partial N}{\partial x} = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is neither homogeneous nor of the form

$yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2y}{-2y} = 1 = f(x)$ say

∴ IF $= e^{\int f(x)dx} = e^{\int 1dx} = e^x$

∴ ① may be rewritten after multiplying by IF as:

$e^x(x^2 - y^2 + 2x)dx - 2e^x y\, dy = 0$.......②

New $M = e^x(x^2 - y^2 + 2x)$, New $N = -2e^x y$

$\frac{\partial M}{\partial y} = -2e^x y$, $\frac{\partial N}{\partial x} = -2e^x y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$$\underset{y\ constant}{\int} e^x(x^2 - y^2 + 2x)\, dx + \int 0\, dy$$

$\Rightarrow (x^2 - y^2 + 2x)e^x - (2x + 2)e^x + (2)e^x = C$

$\Rightarrow (x^2 - y^2)e^x = C$, C is an arbitrary constant

Example 26 Solve the differential equation:

$2ydx + (2x \log x - xy)\, dy = 0$ .....①

Solution: $M = 2y$, $N = 2x \log x - xy$

$\frac{\partial M}{\partial y} = 2$, $\frac{\partial N}{\partial x} = 2(1 + \log x) - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2\log x + y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2\log x + y}{x(2\log x - y)} = -\frac{1}{x} = f(x)$ say

∴ IF $= e^{\int f(x)dx} = e^{\int \frac{-1}{x}dx} = e^{\log x^{-1}} = \frac{1}{x}$

∴ ① may be rewritten after multiplying by IF as:

$\frac{2y}{x}dx + (2\log x - y)\, dy = 0$ ....②

New $M = \frac{2y}{x}$, New $N = 2\log x - y$

$\frac{\partial M}{\partial y} = \frac{2}{x} = \frac{\partial N}{\partial x} = \frac{2}{x}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,∴ ② is an exact differential equation.

Solution of ② is given by:

$$\underset{y\ constant}{\int \frac{2y}{x}\, dx} + \int -y\, dy$$

$\Rightarrow 2y \log x - \frac{y^2}{2} = C$

3.5.4 Integrating Factor (IF) of a Non-Exact Differential Equation

$\boldsymbol{x^a y^b (m_1 ydx + n_1 xdy) + x^c y^d (m_2 ydx + n_2 xdy) = 0}$, where a, b, c, d, m_1, n_1, m_2, n_2 are constants, is given by $x^\alpha y^\beta$, where α and β are connected by the relation $\frac{a + \alpha + 1}{m_1} = \frac{b + \beta + 1}{n_1}$ and $\frac{c + \alpha + 1}{m_2} = \frac{d + \beta + 1}{n_2}$

Example 27 Solve the differential equation:

$(y^2 + 2x^2 y)dx + (2x^3 - xy)dy = 0$ .....①

Solution: $M = y^2 + 2x^2y,\ N = 2x^3 - xy$

$\frac{\partial M}{\partial y} = 2y + 2x^2$, $\frac{\partial N}{\partial x} = 6x^2 - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rewriting ① as $x^2y^0(2ydx + 2xdy) + x^0y^1(ydx - xdy) = 0$ ....②

Comparing with standard form $a = 2,\ b = 0,\ c = 0,\ d = 1,$

$$m_1 = 2,\ n_1 = 2,\ m_2 = 1,\ n_2 = -1$$

$\therefore\ \frac{2+\alpha+1}{2} = \frac{0+\beta+1}{2}$ and $\frac{0+\alpha+1}{1} = \frac{1+\beta+1}{-1}$

$\Rightarrow\ \alpha - \beta = -2$ and $\alpha + \beta = -3$

Solving we get $\alpha = \frac{-5}{2}$ and $\beta = \frac{-1}{2}$

$\therefore \text{IF} = x^{\alpha}y^{\beta} = x^{\frac{-5}{2}}y^{\frac{-1}{2}}$

∴ ① may be rewritten after multiplying by IF as:

$x^{\frac{-5}{2}}y^{\frac{-1}{2}}(y^2 + 2x^2y)dx + x^{\frac{-5}{2}}y^{\frac{-1}{2}}(2x^3 - xy)dy = 0$.......②

$\Rightarrow \left(x^{\frac{-5}{2}}y^{\frac{3}{2}} + 2x^{\frac{-1}{2}}y^{\frac{1}{2}}\right)dx + \left(2x^{\frac{1}{2}}y^{\frac{-1}{2}} - x^{\frac{-3}{2}}y^{\frac{1}{2}}\right)dy = 0$

New $M = x^{\frac{-5}{2}}y^{\frac{3}{2}} + 2x^{\frac{-1}{2}}y^{\frac{1}{2}}$, New $N = 2x^{\frac{1}{2}}y^{\frac{-1}{2}} - x^{\frac{-3}{2}}y^{\frac{1}{2}}$

$\frac{\partial M}{\partial y} = \frac{3}{2}x^{\frac{-5}{2}}y^{\frac{1}{2}} + x^{\frac{-1}{2}}y^{\frac{-1}{2}}$, $\frac{\partial N}{\partial x} = \frac{3}{2}x^{\frac{-5}{2}}y^{\frac{1}{2}} + x^{\frac{-1}{2}}y^{\frac{-1}{2}}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,∴ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant} \left(x^{\frac{-5}{2}}y^{\frac{3}{2}} + 2x^{\frac{-1}{2}}y^{\frac{1}{2}}\right)dx + \int 0\, dy$$

$\Rightarrow\ 4(xy)^{\frac{1}{2}} - \frac{2}{3}\left(\frac{y}{x}\right)^{\frac{3}{2}} = C$, C is an arbitrary constant

Exercise 3.2

Solve the following differential equations:

1. $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$

   Ans. $\langle ax^2 + 2hxy + by^2 + 2gx + 2fy = 0\rangle$

2. $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$

   Ans. $\langle e^{xy^2} + x^4 - y^3 = C\rangle$

3. $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$

Ans. $\langle x + ye^{x^3} = Cy \rangle$

4. $y(2xy + e^x)dx = e^x dy$

Ans. $\langle \frac{e^x}{y} + x^2 = C \rangle$

5. $(y \log x)dx + (x - \log y)dy = 0$

Ans. $\langle (x \log x) - \frac{1}{2}(\log y)^2 = C \rangle$

6. $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0$

Ans. $\langle x^2(x - ay)y = C \rangle$

7. $y(x^2 y^2 + xy + 1)dx + x(x^2 y^2 - xy + 1)dy = 0$

Ans. $\langle 2x^2y^2 + xy\log\frac{x^2}{y} - 2 = Cxy \rangle$

8. $(x^3 y^2 + x)dy + (x^2 y^3 - y)dx = 0$

Ans. $\langle \log\frac{y}{x} + \frac{1}{2}x^2y^2 = C \rangle$

9. $(3y^2 + 2xy)dx - (2xy + x^2)dy = 0$

Ans. $\langle x^3 = Cy(x + y) \rangle$

10. $(2x^2y - xy^2 + y)dx + (x - y)dy = 0$

Ans. $\langle e^{x^2}(2xy - y^2) = C \rangle$

3.6 Linear Differential Equations of Second and Higher Order

A differential equation of the form $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right)=0$ in which the dependent variable $y(x)$ and its derivatives viz. $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc occur in first degree and are not multiplied together is called a Linear Differential Equation.

3.7 Linear Differential Equations (LDE) with Constant Coefficients

A general linear differential equation of n^{th} order with constant coefficients is given by:

$$k_0 \frac{d^ny}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + - - - - - + k_{n-1} \frac{dy}{dx} + k_n y = F(x)$$

where $k's$ are constant and $F(x)$ is a function of x alone or constant.
$\Rightarrow (k_0 D^n + k_1 D^{n-1} + - - - - - + k_{n-1} D + k_n)y = F(x)$

Or $f(D)y = F(x)$, where $D^n \equiv \frac{d^n}{dx^n}$, $D^{n-1} \equiv \frac{d^{n-1}}{dx^{n-1}}$, ..., $D \equiv \frac{d}{dx}$ are called differential operators.

3.8 Solving Linear Differential Equations with Constant Coefficients

Complete solution of equation $f(D)y = F(x)$ is given by $y =$ C.F + P.I.

where C.F. denotes complimentary function and P.I. is particular integral.

When $F(x) = 0$, then solution of equation $f(D)y = 0$ is given by $y =$ C.F

3.8.1 Rules for Finding Complimentary Function (C.F.)

Consider the equation $f(D)y = F(x)$
$\Rightarrow (k_0 D^n + k_1 D^{n-1} + - - - - - + k_{n-1} D + k_n)y = F(x)$

Step 1: Put $D = m$, auxiliary equation (A.E) is given by $f(m) = 0$

$\Rightarrow k_0 m^n + k_1 m^{n-1} + - - - - - + k_{n-1} m + k_n = 0$ ......③

Step 2: Solve the auxiliary equation given by ③

I. If the n roots of A.E. are real and distinct say $m_1, m_2, \ldots m_n$
C.F. = $c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}$

II. If two or more roots are equal i.e. $m_1 = m_2 = \ldots = m_k, k \leq n$
C.F. = $(c_1 + c_2x + c_3x^2 + \cdots + c_kx^{k-1})e^{m_1x} + \cdots + c_ne^{m_nx}$

III. If A.E. has a pair of imaginary roots i.e. $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$

C.F. = $e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3e^{m_3x} + \cdots + c_ne^{m_nx}$

IV. If 2 pairs of imaginary roots are equal i.e. $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$
C.F. = $e^{\alpha x}[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x] + \cdots + c_ne^{m_nx}$

Example 1 Solve the differential equation: $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$

Solution: $\Rightarrow (D^2 - 8D + 15)y = 0$

Auxiliary equation is: $m^2 - 8m + 15 = 0$

$\Rightarrow (m-3)(m-5) = 0$

$\Rightarrow m = 3,5$

C.F. = $c_1e^{3x} + c_2e^{5x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^{3x} + c_2e^{5x}$

Example 2 Solve the differential equation: $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Solution: $\Rightarrow (D^3 - 6D^2 + 11D - 6)y = 0$

Auxiliary equation is: $m^3 - 6m^2 + 11m - 6 = 0$ ....①

By hit and trial $(m-2)$ is a factor of ①

∴① May be rewritten as

$m^3 - 2m^2 - 4m^2 + 8m + 3m - 6 = 0$

$\Rightarrow m^2(m-2) - 4m(m-2) + 3(m-2) = 0$

$\Rightarrow (m^2 - 4m + 3)(m-2) = 0$

$\Rightarrow (m-3)(m-1)(m-2) = 0$

$\Rightarrow m = 1,2,3$

C.F. = $c_1e^x + c_2e^{2x} + c_3e^{3x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^x + c_2e^{2x} + c_3e^{3x}$

Example 3 Solve $(D^4 - 10D^3 + 35D^2 - 50D + 24)y = 0$

Solution: Auxiliary equation is:

$m^4 - 10m^3 + 35m^2 - 50m + 24 = 0$ ....①

By hit and trial $(m - 1)$ is a factor of ①

∴① May be rewritten as

$m^4 - m^3 - 9m^3 + 9m^2 + 26m^2 - 26m - 24m + 24 = 0$

$\Rightarrow m^3(m-1) - 9m^2(m-1) + 26m(m-1) - 24(m-1) = 0$

$\Rightarrow (m-1)(m^3 - 9m^2 + 26m - 24) = 0$ ......②

By hit and trial $(m - 2)$ is a factor of ②

∴② May be rewritten as

$(m-1)(m^3 - 2m^2 - 7m^2 + 14m + 12m - 24) = 0$

$\Rightarrow (m-1)[m^2(m-2) - 7m(m-2) + 12(m-2)] = 0$

$\Rightarrow (m-1)(m^2 - 7m + 12)(m-2) = 0$

$\Rightarrow (m-1)(m-3)(m-4)(m-2) = 0$

$\Rightarrow m = 1,2,3,4$

C.F. = $c_1e^x + c_2e^{2x} + c_3e^{3x} + c_4e^{4x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^x + c_2e^{2x} + c_3e^{3x} + c_4e^{4x}$

Example 4 Solve the differential equation: $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Solution: $\Rightarrow (D^3 + 2D^2 + D)y = 0$

Auxiliary equation is: $m^3 + 2m^2 + m = 0$

$\Rightarrow m(m^2 + 2m + 1) = 0$

$\Rightarrow m(m + 1)^2 = 0$

$\Rightarrow m = 0, -1, -1$

C.F. = $c_1 + (c_2 + c_3x)e^{-x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1 + (c_2 + c_3x)e^{-x}$

Example 5 Solve the differential equation: $\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + y = 0$

Solution: $\Rightarrow (D^4 - 2D^2 + 1)y = 0$

Auxiliary equation is: $m^4 - 2m^2 + 1 = 0$

$\Rightarrow (m^2 - 1)^2 = 0$

$\Rightarrow (m + 1)^2(m - 1)^2 = 0$

$\Rightarrow m = -1, -1, 1, 1$

C.F. = $(c_1 + c_2x)e^{-x} + (c_3 + c_4x)e^x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = (c_1 + c_2x)e^{-x} + (c_3 + c_4x)e^x$

Example 6 Solve the differential equation: $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 0$

Solution: $\Rightarrow (D^3 - 2D + 4)y = 0$

Auxiliary equation is: $m^3 - 2m + 4 = 0$ ....①

By hit and trial $(m + 2)$ is a factor of ①

∴① May be rewritten as

$m^3 + 2m^2 - 2m^2 - 4m + 2m + 4 = 0$

$\Rightarrow m^2(m + 2) - 2m(m + 2) + 2(m + 2) = 0$

$\Rightarrow (m + 2)(m^2 - 2m + 2) = 0$

$\Rightarrow m = -2, 1 \pm i$

C.F. = $c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$

Example 7 Solve the differential equation: $(D^2 - 2D + 5)^2 y = 0$

Solution: Auxiliary equation is: $(m^2 - 2m + 5)^2$.......①

Solving ①, we get

$\Rightarrow m = 1 \pm 2i, 1 \pm 2i$

C.F. = $e^x[(c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = e^x[(c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x]$

Example 8 Solve the differential equation: $(D^2 + 4)^3 y = 0$

Solution: Auxiliary equation is: $(m^2 + 4)^3$.......①

Solving ①, we get

$\Rightarrow m = \pm 2i, \pm 2i, \pm 2i$

C.F. = $(c_1 + c_2 x + c_3 x^2) \cos 2x + (c_4 + c_5 x + c_6 x^2) \sin 2x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = (c_1 + c_2 x + c_3 x^2) \cos 2x + (c_4 + c_5 x + c_6 x^2) \sin 2x$

3.8.2 Shortcut Rules for Finding Particular Integral (P.I.)

Consider the equation $(D)y = F(x)$, $F(x) \neq 0$
$\Rightarrow (k_0 D^n + k_1 D^{n-1} + - - - - - + k_{n-1} D + k_n) y = F(x)$

Then P.I = $\frac{1}{f(D)} F(x)$, Clearly P.I. = 0 if $F(x) = 0$

Case I: When $F(x) = e^{ax}$

Use the rule P.I $= \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$, $f(a) \neq 0$

In case of failure i.e. if $f(a) = 0$

P.I = $x\frac{1}{f'(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax}$, $f'(a) \neq 0$

If $f'(a) = 0$, P.I. = $x^2\frac{1}{f''(a)}e^{ax}$, $f''(a) \neq 0$ and so on

Example 9 Solve the differential equation: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = e^{2x}$

Solution: $\Rightarrow (D^2 - 2D + 10)y = e^{2x}$

Auxiliary equation is: $m^2 - 2m + 10 = 0$

$\Rightarrow m = 1 \pm 3i$

C.F. = $e^x(c_1 \cos 3x + c_2 \sin 3x)$

P.I. = $\frac{1}{f(D)} F(x) = \frac{1}{f(D)}e^{2x} = \frac{1}{f(2)}e^{2x}$, by putting $D = 2$

$= \frac{1}{2^2 - 2(2) + 10}e^{2x} = \frac{1}{10}e^{2x}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = e^x(c_1 \cos 3x + c_2 \sin 3x) + \frac{1}{10}e^{2x}$

Example 10 Solve the differential equation: $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^x$

Solution: $\Rightarrow (D^2 + D - 2)y = e^x$

Auxiliary equation is: $m^2 + m - 2 = 0$

$\Rightarrow (m + 2)(m - 1) = 0$

$\Rightarrow m = -2, 1$

C.F. = $c_1e^{-2x} + c_2e^x$

P.I. = $\frac{1}{f(D)} F(x) = \frac{1}{f(D)}e^x$, putting $D = 1, f(1) = 0$

$\therefore$P.I = $x\frac{1}{f'(D)}e^x$ $\qquad$ $\because$ P.I. = $x\frac{1}{f'(a)}e^{ax}$ if $f(a) = 0$

$\Rightarrow$ P.I. = $x\frac{1}{2D+1}e^x = \frac{1}{f'(1)}e^x$, $f'(1) \neq 0$

$\Rightarrow$ P.I. = $\frac{xe^x}{3}$

Complete solution is: $y = $ C.F. + P.I

$\Rightarrow y = c_1e^{-2x} + c_2e^{x} + \frac{xe^x}{3}$

Example 11 Solve the differential equation: $\frac{d^2y}{dx^2} - 4y = \sinh(2x+1) + 4^x$

Solution: $\Rightarrow (D^2 - 4)y = \sinh(2x+1) + 4^x$

Auxiliary equation is: $m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

C.F. $= c_1e^{2x} + c_2e^{-2x}$

P.I. $= \frac{1}{f(D)} F(x)$

$= \frac{1}{f(D)} (\sinh(2x+1) + 4^x)$

$= \frac{1}{D^2-4}\left(\frac{e^{(2x+1)} - e^{-(2x+1)}}{2}\right) + \frac{1}{D^2-4}\left(e^{x\log 4}\right)$

$\because \sinh x = \frac{e^x - e^{-x}}{2}$ and $4^x = e^{x\log 4}$

$= \frac{e}{2}\frac{1}{D^2-4}e^{2x} - \frac{e^{-1}}{2}\frac{1}{D^2-4}e^{-2x} + \frac{1}{D^2-4}e^{x\log 4}$

Putting $D = 2, -2$ and $\log 4$ in the three terms respectively

$f(2) = 0$ and $f(-2) = 0$ for first two terms

$\therefore$ P.I. $= \frac{e}{2}x\frac{1}{2D}e^{2x} - \frac{e^{-1}}{2}x\frac{1}{2D}e^{-2x} + \frac{1}{(log4)^2-4}e^{x\log 4}$

$\because$ P.I. $= x\frac{1}{f'(a)}e^{ax}$ if $f(a) = 0$

Now putting $D = 2, -2$ in first two terms respectively

$\Rightarrow$P.I. $= \frac{ex}{8}e^{2x} + \frac{e^{-1}x}{8}e^{-2x} + \frac{4^x}{(log4)^2-4}$ $\quad\because e^{x\log 4} = 4^x$

$\Rightarrow$P.I. $= \frac{x}{4}\left(\frac{e^{(2x+1)} + e^{-(2x+1)}}{2}\right) + \frac{4^x}{(log4)^2-4}$

$\Rightarrow$P.I. $= \frac{x}{4}\cosh(2x+1) + \frac{4^x}{(log4)^2-4}$ $\quad\because \cosh x = \frac{e^x + e^{-x}}{2}$

Complete solution is: $y = $ C.F. + P.I

$$\Rightarrow y = c_1e^{2x} + c_2e^{-2x} + \frac{x}{4}\cosh(2x+1) + \frac{4^x}{(log4)^2 - 4}$$

Case II: When $F(x) = Sin\,(ax+b) or\ Cos\,(ax+b)$

If $F(x) = Sin\,(ax+b)\ or\ Cos\,(ax+b)$, put $D^2 = -a^2$,

$D^3 = D^2D = -a^2D, D^4 = (D^2)^2 = a^4, \ldots\ldots$

This will form a linear expression in D in the denominator. Now rationalize the denominator to substitute $D^2 = -a^2$. Operate on the numerator term by term by taking $D \equiv \frac{d}{dx}$

In case of failure i.e. if $f(-a^2) = 0$

P.I. $= x\frac{1}{f'(-a^2)}\ Sin\,(ax+b)\ or\ Cos\,(ax+b), f'(-a^2) \neq 0$

If $f'(-a^2) = 0$, P.I.= $x^2\frac{1}{f''(-a^2)}\ Sin\,(ax+b)\ or\ Cos\,(ax+b), f''(-a^2) \neq 0$

Example 12 Solve the differential equation: $(D^2 + D - 2)y = \sin x$

Solution: Auxiliary equation is: $m^2 + m - 2 = 0$

$\Rightarrow (m+2)(m-1) = 0$

$\Rightarrow\ m = -2,1$

C.F. = $c_1e^{-2x} + c_2e^x$

P.I. $= \frac{1}{f(D)}\ F(x) = \frac{1}{f(D)}\sin x = \frac{1}{D^2+D-2}\sin x$

putting $D^2 = -1^2 = -1$

P.I = $\frac{1}{D-3}\sin x = \frac{D+3}{D^2-9}\sin x$, Rationalizing the denominator

$= \frac{(D+3)\sin x}{-10}$, Putting $D^2 = -1$

$\therefore$ P. I. $= \frac{-1}{10}(D\,sin\,x + 3\ \sin x)$

$= \frac{-1}{10}(\cos x + 3\ \sin x)$

Complete solution is: $y = $ C.F. + P.I

$\Rightarrow y = c_1 e^{-2x} + c_2 e^{x} - \frac{1}{10}(\cos x + 3 \sin x)$

Example 13 Solve the differential equation: $(D^2 + 2D + 1)y = cos^2 x$

Solution: Auxiliary equation is: $m^2 + 2m + 1 = 0$

$(m + 1)^2 = 0$

$\Rightarrow m = -1, -1$

C.F. = $e^{-x}(c_1 + c_2 x)$

P.I. = $\frac{1}{f(D)} F(x) = \frac{1}{f(D)} cos^2 x = \frac{1}{D^2+2D+1}\left(\frac{1+\cos 2x}{2}\right)$

$= \frac{1}{2}\frac{1}{D^2+2D+1} e^{0x} + \frac{1}{2}\frac{1}{D^2+2D+1} \cos 2x$

Putting $D = 0$ in the 1st term and $D^2 = -2^2 = -4$ in the 2nd term

P.I = $\frac{1}{2} + \frac{1}{2}\frac{1}{2D-3} \cos 2x$

$= \frac{1}{2} + \frac{1}{2}\frac{2D+3}{4D^2-3^2} \cos 2x$, Rationalizing the denominator

$= \frac{1}{2} + \frac{1}{2}\frac{(2D+3)\cos 2x}{-25}$, Putting $D^2 = -4$

$\therefore$ P. I. = $\frac{1}{2} - \frac{1}{50}(-4 \sin 2x + 3 \cos 2x)$

Now $y = $ C.F. + P.I

$\Rightarrow y = e^{-x}(c_1 + c_2 x) + \frac{1}{2} - \frac{1}{50}(-4 \sin 2x + 3 \cos 2x)$

Example 14 Solve the differential equation: $(D^2 + 9)y = \sin 2x \cos x$

Solution: Auxiliary equation is: $m^2 + 9 = 0$

$\Rightarrow m = \pm 3i$

C.F. = $c_1 \cos 3x + c_2 \sin 3x$

P.I. = $\frac{1}{f(D)} F(x) = \frac{1}{f(D)} \sin 2x \cos x = \frac{1}{2}\frac{1}{D^2+9}(\sin 3x + \sin x)$

$= \frac{1}{2}\frac{1}{D^2+9}\sin 3x + \frac{1}{2}\frac{1}{D^2+9}\sin x$

Putting $D^2 = -9$ in the 1st term and $D^2 = -1$ in the 2nd term

We see that $f(D^2 = -9) = 0$ for the 1st term

$\therefore$ P.I. $= \frac{1}{2}\, x \frac{1}{2D}\sin 3x + \frac{1}{2}\frac{1}{8}\sin x$

$\because$P.I. $= x\frac{1}{f'(-a^2)}\, Sin\,(ax+b)\,,\, f'(-a^2) \neq 0$

$\Rightarrow$P.I. $= -\frac{x}{12}\cos 3x + \frac{1}{16}\sin x$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{12}\cos 3x + \frac{1}{16}\sin x$

Case III: When $F(x) = x^n$, n *is a positive integer*

P.I $= \frac{1}{f(D)} F(x) = \frac{1}{f(D)} x^n$

1. Take the lowest degree term common from $f(D)$ to get an expression of the form $[1 \pm \phi(D)]$ in the denominator and take it to numerator to become $[1 \pm \phi(D)]^{-1}$
2. Expand $[1 \pm \phi(D)]^{-1}$ using binomial theorem up to n^{th} degree as $(n+1)^{th}$ derivative of x^n is zero
3. Operate on the numerator term by term by taking $D \equiv \frac{d}{dx}$

Following expansions will be useful to expand $[1 \pm \phi(D)]^{-1}$in ascending powers of D

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
- $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$
- $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

Example 15 Solve the differential equation: $\frac{d^2y}{dx^2} - y = 5x - 2$

Solution: $\Rightarrow (D^2 - 1)y = 5x - 2$

Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow\ m = \pm 1$

C.F. = $c_1 e^x + c_2 e^{-x}$

P.I. = $\frac{1}{f(D)}\ F(x)\ = \frac{1}{D^2-1}(5x-2)$

$= \frac{1}{-(1-D^2)}(5x-2)$

$= -(1-D^2)^{-1}(5x-2)$

$= -[1 + D^2 + \cdots](5x-2)$

$= -(5x-2)$

∴P.I = $-5x+2$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ =\ c_1 e^x + c_2 e^{-x} - 5x + 2$

Example 16 Solve the differential equation: $(D^4 + 4D^2)y = x^2 + 1$

Solution: Auxiliary equation is: $m^4 + 4m^2 = 0$

$\Rightarrow\ m^2(m^2+4) = 0$

$\Rightarrow m = 0, 0, \pm 2i$

C.F. = $(c_1 + c_2 x) + (c_3 \cos 2x + c_4 \sin 2x)$

P.I. = $\frac{1}{f(D)}\ F(x)\ = \frac{1}{D^4+4D^2}(x^2+1)$

$= \frac{1}{D^4+4D^2}(x^2+1)$

$= \frac{1}{4D^2\left(1+\frac{D^2}{4}\right)}(x^2+1)$

$= \frac{1}{4D^2}\left(1+\frac{D^2}{4}\right)^{-1}(x^2+1)$

$= \frac{1}{4D^2}\left[1-\frac{D^2}{4}+\cdots\right](x^2+1)$

$= \frac{1}{4D^2}\left(x^2+1-\frac{1}{2}\right)$

$= \frac{1}{4D^2}\left(x^2 + \frac{1}{2}\right)$

$= \frac{1}{4D}\int\left(x^2 + \frac{1}{2}\right)dx$

$= \frac{1}{4D}\left(\frac{x^3}{3} + \frac{x}{2}\right)$

$= \frac{1}{4}\int\left(\frac{x^3}{3} + \frac{x}{2}\right)dx$

$\therefore$ P.I $= \frac{1}{4}\left(\frac{x^4}{12} + \frac{x^2}{4}\right)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = (c_1 + c_2x) + (c_3 \cos 2x + c_4 \sin 2x) + \frac{1}{4}\left(\frac{x^4}{12} + \frac{x^2}{4}\right)$

Example 17 Solve the differential equation: $(D^2 - 6D + 9)y = 1 + x + x^2$

Solution: Auxiliary equation is: $m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0$

$\Rightarrow m = 3,3$

C.F. $= e^{3x}(c_1 + c_2x)$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{D^2-6D+9}(1 + x + x^2)$

$= \frac{1}{9\left(1-\frac{2D}{3}+\frac{D^2}{9}\right)}(1 + x + x^2)$

$= \frac{1}{9}\left(1 - \left(\frac{2D}{3} - \frac{D^2}{9}\right)\right)^{-1}(1 + x + x^2)$

$= \frac{1}{9}\left[1 + \left(\frac{2D}{3} - \frac{D^2}{9}\right) + \left(\frac{2D}{3} - \frac{D^2}{9}\right)^2 + \cdots\right](1 + x + x^2)$

$= \frac{1}{9}\left[1 + \frac{2D}{3} - \frac{D^2}{9} + \frac{4D^2}{9} + \cdots\right](1 + x + x^2)$

$= \frac{1}{9}\left[1 + \frac{2D}{3} + \frac{D^2}{3} + \cdots\right](1 + x + x^2)$

$= \frac{1}{9}\left(1 + x + x^2 + \frac{2}{3} + \frac{4x}{3} + \frac{2}{3}\right)$

$\therefore$ P.I $= \frac{1}{9}\left(\frac{7}{3} + \frac{7x}{3} + x^2\right)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = e^{3x}(c_1 + c_2x) + \frac{1}{9}\left(\frac{7}{3} + \frac{7x}{3} + x^2\right)$

Case IV: When $F(x) = e^{ax}g(x)$, where $g(x)$ is any function of x

Use the rule: $\frac{1}{f(D)} e^{ax}g(x) = e^{ax}\left(\frac{1}{f(D+a)} g(x)\right)$

Example 18 Solve the differential equation: $(D^2 + 2)y = x^2e^{3x}$

Solution: Auxiliary equation is: $m^2 + 2 = 0$

$\Rightarrow m^2 = -2$

$\Rightarrow m = \pm\sqrt{2}i$

C.F. = $(c_1 \cos(\sqrt{2}\,x) + c_2 \sin(\sqrt{2}\,x))$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+2} x^2e^{3x}$

$$= e^{3x} \frac{1}{(D+3)^2+2} x^2$$

$$= e^{3x} \frac{1}{D^2+6D+11} x^2$$

$$= \frac{e^{3x}}{11} \frac{1}{\left(1+\frac{6D}{11}+\frac{D^2}{11}\right)} x^2$$

$$= \frac{e^{3x}}{11}\left(1 + \left(\frac{6D}{11} + \frac{D^2}{11}\right)\right)^{-1} x^2$$

$$= \frac{e^{3x}}{11}\left[1 - \left(\frac{6D}{11} + \frac{D^2}{11}\right) + \left(\frac{6D}{11} + \frac{D^2}{11}\right)^2 + \cdots\right]x^2$$

$$= \frac{e^{3x}}{11}\left[1 - \frac{6D}{11} - \frac{D^2}{11} + \frac{36D^2}{121} + \cdots\right]x^2$$

$$= \frac{e^{3x}}{11}\left[1 - \frac{6D}{11} + \frac{25D^2}{121} + \cdots\right]x^2$$

$$= \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$$

$\therefore\ P.I = \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = (c_1 \cos(\sqrt{2}\,x) + c_2 \sin(\sqrt{2}\,x)) + \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$

Example 19 Solve the differential equation: $(D^3 + 1)y = e^{2x} \sin x$

Solution: Auxiliary equation is: $m^3 + 1 = 0$

$\Rightarrow m^3 = -1$

$\Rightarrow m = -1, \frac{1 \pm \sqrt{3}i}{2}$

C.F. $= c_1 e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^3+1} e^{2x} \sin x$

$= e^{2x} \frac{1}{(D+2)^3+1} \sin x$

$= e^{2x} \frac{1}{D^3+6D^2+12D+9} \sin x$

$= e^{2x} \frac{1}{-D-6+12D+9} \sin x$, Putting $D^2 = -1$

$= e^{2x} \frac{1}{11D+3} \sin x$

$= e^{2x} \frac{11D-3}{121D^2-9} \sin x$, Rationalizing the denominator

$= -\frac{e^{2x}}{130}(11D - 3) \sin x$, Putting $D^2 = -1$

$\therefore$ P.I $= -\frac{e^{2x}}{130}(11 \cos x - 3 \sin x)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

$- \frac{e^{2x}}{130}(11 \cos x - 3 \sin x)$

Example 20 Solve the differential equation: $\frac{d^2y}{dx^2} - 4y = x\sinh x$

Solution: $\Rightarrow (D^2 - 4)y = x\sinh x$

Auxiliary equation is: $m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

C.F. $= c_1e^{2x} + c_2e^{-2x}$

P.I. $= \frac{1}{f(D)} F(x)$

$= \frac{1}{f(D)}(x\sinh x)$

$= \frac{1}{D^2-4}\left(x\frac{e^x - e^{-x}}{2}\right)$ $\qquad \because \sinh x = \frac{e^x - e^{-x}}{2}$

$= \frac{1}{D^2-4}\left(x\frac{e^x}{2} - x\frac{e^{-x}}{2}\right)$

$= \frac{e^x}{2}\frac{1}{(D+1)^2-4}x - \frac{e^{-x}}{2}\frac{1}{(D-1)^2-4}x$

$= \frac{e^x}{2}\frac{1}{(D^2+2D-3)}x - \frac{e^{-x}}{2}\frac{1}{D^2-2D-3}x$

$= \frac{e^x}{2}\frac{1}{-3\left(1-\frac{D^2}{3}-\frac{2D}{3}\right)}x - \frac{e^{-x}}{2}\frac{1}{-3\left(1-\frac{D^2}{3}+\frac{2D}{3}\right)}x$

$= -\frac{e^x}{6}\left[1-\left(\frac{D^2}{3}+\frac{2D}{3}\right)\right]^{-1}x + \frac{e^{-x}}{6}\left[1-\left(\frac{D^2}{3}-\frac{2D}{3}\right)\right]^{-1}x$

$= -\frac{e^x}{6}\left(1+\frac{2D}{3}\right)x + \frac{e^{-x}}{6}\left(1-\frac{2D}{3}\right)x$

$= -\frac{e^x}{6}\left(x+\frac{2}{3}\right) + \frac{e^{-x}}{6}\left(x-\frac{2}{3}\right)$

$= -\frac{x}{3}\left(\frac{e^x - e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x + e^{-x}}{2}\right)$

$\therefore$ P.I. $= -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1e^{2x} + c_2e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$

Example 21 Solve the differential equation: $(D^2+1)y = x^2 \sin 2x$

Solution: Auxiliary equation is: $m^2+1=0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = \pm i$

C.F. $= c_1 \cos x + c_2 \sin x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+1} x^2 \sin 2x$

$= \text{Imaginary part of } \frac{1}{D^2+1} x^2 e^{i2x}$

Now $\frac{1}{D^2+1} x^2 e^{i2x} = e^{i2x} \frac{1}{(D+2i)^2+1} x^2$

$$= e^{i2x} \frac{1}{D^2+4i^2+4iD+1} x^2$$

$$= e^{i2x} \frac{1}{D^2+4iD-3} x^2$$

$$= e^{i2x} \frac{1}{-3\left(1-\frac{D^2}{3}-\frac{4iD}{3}\right)} x^2$$

$$= \frac{-e^{i2x}}{3}\left[1-\left(\frac{D^2}{3}+\frac{4iD}{3}\right)\right]^{-1} x^2$$

$$= \frac{-e^{i2x}}{3}\left[1+\left(\frac{D^2}{3}+\frac{4iD}{3}\right)+\left(\frac{D^2}{3}+\frac{4iD}{3}\right)^2\right] x^2$$

$$= \frac{-e^{i2x}}{3}\left[1+\frac{D^2}{3}+\frac{4iD}{3}+\frac{16i^2D^2}{9}\right] x^2$$

$$= \frac{-e^{i2x}}{3}\left[1-\frac{13D^2}{9}+\frac{4iD}{3}\right] x^2$$

$$= \frac{-e^{i2x}}{3}\left[x^2-\frac{26}{9}+i\frac{8x}{3}\right]$$

$$= -\frac{1}{3}(\cos 2x + i \sin 2x)\left[x^2-\frac{26}{9}+i\frac{8x}{3}\right]$$

$\therefore$ P.I. $= \text{Imaginary part of } \frac{1}{D^2+1} x^2 e^{i2x} = -\frac{1}{3}\left(\frac{8x}{3}\cos 2x + \left(x^2-\frac{26}{9}\right)\sin 2x\right)$

$$= -\frac{8x}{9}\cos 2x + \frac{1}{27}(26-9x^2)\sin 2x$$

Complete solution is: $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 \cos x + c_2 \sin x - \frac{8x}{9}\cos 2x + \frac{1}{27}(26 - 9x^2)\sin 2x$$

Example 22 Solve the differential equation: $(D^2 - 4D + 4)y = x^2 e^{2x} \sin 2x$

Solution: Auxiliary equation is: $m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2$

$\Rightarrow m = 2,2$

C.F. $= (c_1 + c_2 x)e^{2x}$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{D^2 - 4D + 4} x^2 e^{2x} \sin 2x$$

$$= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^2 \sin 2x$$

$$= e^{2x} \frac{1}{D^2} x^2 \sin 2x$$

$$= e^{2x} \frac{1}{D} \int x^2 \sin 2x \, dx$$

$$= e^{2x} \frac{1}{D}\left[(x^2)\left(\frac{-\cos 2x}{2}\right) - (2x)\left(\frac{-\sin 2x}{4}\right) + (2)\left(\frac{\cos 2x}{8}\right)\right]$$

$$= e^{2x} \frac{1}{D}\left[-\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x\right]$$

$$= e^{2x}\left[-\frac{1}{2}\int x^2 \cos 2x + \frac{1}{2}\int x \sin 2x \, dx + \frac{1}{4}\int \cos 2x \, dx\right]$$

$$= e^{2x}\left[-\frac{1}{2}\left[(x^2)\left(\frac{\sin 2x}{2}\right) - (2x)\left(\frac{-\cos 2x}{4}\right) + (2)\left(\frac{-\sin 2x}{8}\right)\right] + \right.$$

$12x{-}\cos 2x2{-}1{-}\sin 2x4{+}14\sin 2x2$

$$\therefore \text{P.I.} = e^{2x}\left[\frac{-x^2}{4}\sin 2x - \frac{x}{2}\cos 2x + \frac{3}{8}\sin 2x\right]$$

Complete solution is: $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = (c_1 + c_2 x)e^{2x} + e^{2x}\left[\frac{-x^2}{4}\sin 2x - \frac{x}{2}\cos 2x + \frac{3}{8}\sin 2x\right]$$

Case V: When $F(x) = x\, g(x)$, where $g(x)$ is any function of x

Use the rule: $\frac{1}{f(D)}\left(x\, g(x)\right) = x \frac{1}{f(D)} g(x) + \left(\frac{d}{dD}\frac{1}{f(D)}\right) g(x)$

Example 23 Solve the differential equation: $(D^2+9)y = x\cos x$

Solution: Auxiliary equation is: $m^2+9=0$

$\Rightarrow m^2 = -9$

$\Rightarrow m = \pm 3i$

C.F. = $(c_1\cos 3x + c_2\sin 3x)$

P.I. $= \frac{1}{f(D)}\, F(x) \; = \frac{1}{D^2+9}\, x\cos x$

$= x\frac{1}{D^2+9}\cos x + \frac{-2D}{(D^2+9)^2}\cos x$

$= x\frac{1}{-1+9}\cos x + \frac{-2D}{(-1+9)^2}\cos x,$ Putting $D^2 = -1$

$= \frac{x\cos x}{8} - \frac{2D\cos x}{64}$

$= \frac{x\cos x}{8} - \frac{2D\cos x}{64}$

$\therefore$ P.I. $= \frac{x\cos x}{8} + \frac{\sin x}{32}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1\cos 3x + c_2\sin 3x + \frac{x\cos x}{8} + \frac{\sin x}{32}$

Example 24 Solve the differential equation:

$$(D^2-1)y = x\sin x + (1+x^2)e^x$$

Solution: Auxiliary equation is: $m^2-1=0$

$\Rightarrow m = \pm 1$

C.F. = $c_1e^x + c_2e^x$

P.I. $= \frac{1}{f(D)}\, F(x) \; = \frac{1}{D^2-1}\,(x\sin x + (1+x^2)e^x)$

$= \frac{1}{D^2-1}x\sin x + \frac{1}{D^2-1}(1+x^2)e^x$

Now $\frac{1}{D^2-1}x\sin x = = x\frac{1}{D^2-1}\sin x + \frac{-2D}{(D^2-1)^2}\sin x$

$$= x\frac{1}{-1-1}\sin x + \frac{-2D}{(-1-1)^2}\sin x, \qquad \text{Putting } D^2 = -1$$

$$= -\frac{1}{2}(x\sin x + \cos x)$$

Also $\frac{1}{D^2-1}(1+x^2)e^x = e^x\frac{1}{(D+1)^2-1}(1+x^2)$

$$= e^x\frac{1}{D^2+2D}(1+x^2)$$

$$= e^x\frac{1}{2D\left(1+\frac{D}{2}\right)}(1+x^2)$$

$$= e^x\frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1}(1+x^2)$$

$$= e^x\frac{1}{2D}\left[1-\frac{D}{2}+\frac{D^2}{4}\right](1+x^2)$$

$$= e^x\frac{1}{2D}\left[1+x^2-x+\frac{1}{2}\right]$$

$$= e^x\frac{1}{2D}\left[x^2-x+\frac{3}{2}\right]$$

$$= \frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}\right]$$

$\therefore$ P.I. $= -\frac{1}{2}(x\sin x + \cos x) + \frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}\right]$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1e^x + c_2e^x - \frac{1}{2}(x\sin x + \cos x) + \frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}\right]$

Case VI: When $F(x)$ is any general function of x not covered in shortcut methods I to V above

Resolve $f(D)$ into partial fractions and use the rule:

$$\frac{1}{D+a}F(x) = e^{-ax}\int e^{ax}F(x)\,dx$$

Example 25 Solve the differential equation: $(D^2+3D+2)y = e^{e^x}$

Solution: Auxiliary equation is: $m^2+3m+2=0$

$\Rightarrow (m+1)(m+2) = 0$

$\Rightarrow m = -1, -2$

C.F. $= c_1 e^{-x} + c_2 e^{-2x}$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+3D+2} e^{e^x}$

$= \frac{1}{(D+1)(D+2)} e^{e^x}$

$= \left(\frac{1}{(D+1)} - \frac{1}{(D+2)}\right) e^{e^x}$

$= e^{-x} \int e^x e^{e^x} dx - e^{-2x} \int e^{2x} e^{e^x} dx$

$= e^{-x} \int D e^{e^x} dx - e^{-2x} \int e^x D e^{e^x} dx$

$= e^{-x} e^{e^x} - e^{-2x}\left[e^x e^{e^x} - \int e^x e^{e^x} dx\right]$, Integrating 2nd term by parts

$= e^{-x} e^{e^x} - e^{-2x}\left[e^x e^{e^x} - \int D e^{e^x} dx\right]$

$= e^{-x} e^{e^x} - e^{-2x}\left[e^x e^{e^x} - e^{e^x}\right]$

$\therefore$ P.I. $= e^{-2x} e^{e^x}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$

Example 26 Solve the differential equation: $(D^2 + 4)y = \tan 2x$

Solution: Auxiliary equation is: $m^2 + 4 = 0$

$\Rightarrow m = \pm 2i$

C.F. $= c_1 \cos 2x + c_2 \sin 2x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+4} \tan 2x$

$= \frac{1}{(D-2i)(D+2i)} \tan 2x$

$= \frac{1}{4i}\left(\frac{1}{(D-2i)} - \frac{1}{(D+2i)}\right) \tan 2x$

P.I. $= \frac{1}{4i}\left(\frac{1}{D-2i} \tan 2x\right) - \frac{1}{4i}\left(\frac{1}{D+2i} \tan 2x\right)$ ....①

Now $\frac{1}{D-2i}\tan 2x = e^{2ix}\int e^{-2ix}\tan 2x\,dx$

$$= e^{2ix}\int(\cos 2x - i\sin 2x)\tan 2x\,dx$$

$$= e^{2ix}\int(\sin 2x - i\frac{\sin^2 2x}{\cos 2x})\,dx$$

$$= e^{2ix}\int\left(\sin 2x - i\frac{1-\cos^2 2x}{\cos 2x}\right)dx$$

$$= e^{2ix}\int(\sin 2x - i\sec 2x + i\cos 2x)\,dx$$

$$= e^{2ix}\left(-\frac{1}{2}\cos 2x - \frac{i}{2}log|\sec 2x + \tan 2x| + \frac{i}{2}\sin 2x\right)$$

$$\therefore \frac{1}{D-2i}\tan 2x = e^{2ix}\left(-\frac{1}{2}e^{-2ix} - \frac{i}{2}log|\sec 2x + \tan 2x|\right) \quad ...②$$

Replacing i by $-i$

$$\frac{1}{D+2i}\tan 2x = e^{-2ix}\left(-\frac{1}{2}e^{2ix} + \frac{i}{2}log|\sec 2x + \tan 2x|\right) \quad ...③$$

Using ②and ③ in ①

$$\text{P.I.} = \frac{1}{4i}\left[e^{2ix}\left(-\frac{1}{2}e^{-2ix} - \frac{i}{2}log|\sec 2x + \tan 2x|\right)\right]$$

$$-\frac{1}{4i}\left[e^{-2ix}\left(-\frac{1}{2}e^{2ix} + \frac{i}{2}log|\sec 2x + \tan 2x|\right)\right]$$

$$= \frac{1}{4i}\left[-\frac{1}{2} - \frac{i}{2}e^{2ix}log|\sec 2x + \tan 2x| + \frac{1}{2} - \frac{i}{2}e^{-2ix}log|\sec 2x + \tan 2x|\right]$$

$$= \frac{1}{4i}\left[-i\frac{e^{2ix}+e^{-2ix}}{2}log|\sec 2x + \tan 2x|\right]$$

$$\therefore \text{P.I} = -\frac{1}{4}[\cos 2x\, log|\sec 2x + \tan 2x|]$$

Complete solution is: $y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}[\cos 2x\, log|\sec 2x + \tan 2x|]$$

Exercise 3.3

Solve the following differential equations:

1. $(D^3 + D^2 - 5D + 3)y = 0$ Ans. $y = (c_1x + c_2)e^x + c_3e^{-3x}$

2. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x}$ Ans. $y = c_1e^{2x} + c_2e^{3x} + e^{3x}(x-1)$

3. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^x \cosh 2x$

Ans. $y = c_1e^{-3x} + c_2e^{2x} + \frac{1}{12}e^{3x} - \frac{1}{12}e^{-x}$

4. $(D-1)^2(D^2+1)^2y = e^x$

Ans. $y = (c_1x + c_2)e^x + (c_3x + c_4)\cos x + (c_5x + c_6)\sin x + \frac{x^2}{8}e^x$

5. $(D^2 - 6D + 9)y = x^2 + 2e^{2x}$

Ans. $y = (c_1x + c_2)e^{3x} + \frac{1}{9}\left(x^2 + \frac{4x}{8} + \frac{2}{3}\right) + 2e^{2x}$

6. $(D^2 + D - 2)y = x + \sin x$

Ans. $y = c_1e^{-2x} + c_2e^x - \frac{1}{4}(2x+1) - \frac{1}{10}(\cos x + 3\sin x)$

7. $(D^2 + D)y = (1+e^x)^{-1}$

Ans. $y = c_1 + c_2e^{-x} + x - (1+e^{-x})\log(1+e^x)$

8. $(D^2 + 5D + 6)y = e^{-2x}sec^2x\,(1+2\tan x)$

Ans. $y = c_1e^{-2x} + c_2e^{-3x} + e^{-2x}(\tan x - 1)$

9. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 12y = (x-1)e^{2x}$

Ans. $y = c_1e^{2x} + c_2e^{-6x} + \frac{e^{2x}}{8}\left(\frac{x^2}{2} - \frac{9x}{8}\right)$

10. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{3x}$, given $y = 1, \frac{dy}{dx} = -1$ when $x = 0$

Ans. $y = -\frac{1}{2}e^x - 2e^{2x} + 2x + 3 + \frac{e^{3x}}{2}$

3.9 Differential Equations Reducible to Linear Form with Constant Coefficients

Some special type of homogenous and non homogeneous linear differential equations with variable coefficients after suitable substitutions can be reduced to linear differential equations with constant coefficients.

3.9.1 Cauchy's Linear Differential Equation

The differential equation of the form:

$$k_0\, x^n \frac{d^ny}{dx^n} + k_1\, x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + - - - - - + k_{n-1}\, x\frac{dy}{dx} + k_n\, y = F(x)$$

is called Cauchy's linear equation and it can be reduced to linear differential equations with constant coefficients by following substitutions:

$x = e^t \quad \Rightarrow \log x = t$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x}$

$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dt} = Dy$, where $D \equiv \frac{d}{dt}$

Similarly $x^2\frac{d^2y}{dx^2} = D(D-1)y$, $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$ and so on.

Example 27 Solve the differential equation:

$x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 13\cos(\log x),\ x > 0$ ....①

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y$ and $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$

∴① May be rewritten as

$(D(D-1)(D-2) + 3D(D-1) + D + 8)y = 13\cos t$

$\Rightarrow (D^3 + 8)y = 13\cos t \ ,\ D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^3 + 8 = 0$

$\Rightarrow (m+2)(m^2 - 2m + 2) = 0$

$\Rightarrow m = -2, 1 \pm \sqrt{3}i$

C.F. $= c_1e^{-2t} + e^t\left(c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t\right)$

$= \frac{c_1}{x^2} + x\left(c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x)\right)$

P.I. $= \frac{1}{f(D)}F(x) = 13\frac{1}{D^3+8}\cos t$

$= 13\frac{1}{-D+8}\cos t$, Putting $D^2 = -1$

$= 13\frac{(8+D)}{64-D^2}\cos t = 13\frac{(8+D)}{65}\cos t$ Putting $D^2 = -1$

$\therefore$ P.I. $= \frac{1}{5}(8\cos t + D\cos t)$

$= \frac{1}{5}(8\cos t - \sin t)$

$= \frac{1}{5}(8\cos(\log x) - \sin(\log x))$

Complete solution is: $y =$ C.F. + P.I

$$\Rightarrow y = \frac{c_1}{x^2} + x\left(c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x\right) +$$

$$\frac{1}{5}(8\cos(\log x) - \sin(\log x))$$

Example 28 Solve the differential equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = \frac{x^3}{1+x^2} \quad ....①$$

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t$ $\therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy, x^2\frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$① May be rewritten as

$(D(D-1) + D - 1)y = \frac{e^{3t}}{1+e^{2t}}$

$\Rightarrow (D^2 - 1)y = \frac{e^{3t}}{1+e^{2t}}, \; D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

C.F. $= c_1e^{-t} + c_2e^t$

$= \frac{c_1}{x} + c_2x$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{D^2-1}\frac{e^{3t}}{1+e^{2t}}$

$= \frac{1}{(D-1)(D+1)}\frac{e^{3t}}{1+e^{2t}} = \frac{1}{2}\left(\frac{1}{(D-1)} - \frac{1}{(D+1)}\right)\frac{e^{3t}}{1+e^{2t}}$

$= \frac{1}{2}\left(\frac{1}{(D-1)}\frac{e^{3t}}{1+e^{2t}} - \frac{1}{(D+1)}\frac{e^{3t}}{1+e^{2t}}\right)$

$= \frac{1}{2}\left(e^{t}\int e^{-t}\frac{e^{3t}}{1+e^{2t}}dt - e^{-t}\int e^{t}\frac{e^{3t}}{1+e^{2t}}dt\right) \because \frac{1}{D+a}F(x) = e^{-ax}\int e^{ax}F(x)\,dx$

$= \frac{1}{2}\left(e^{t}\int \frac{e^{2t}}{1+e^{2t}}dt - e^{-t}\int \frac{e^{4t}}{1+e^{2t}}dt\right)$

Put $e^{2t} = u \Rightarrow 2e^{2t}dt = du$

$\therefore$ P.I $= \frac{1}{4}\left(e^{t}\int \frac{1}{1+u}du - e^{-t}\int \frac{u}{1+u}du\right)$

$= \frac{1}{4}\left(e^{t}\log(1+u) - e^{-t}\int \frac{1+u-1}{1+u}du\right)$

$= \frac{1}{4}\left(e^{t}\log(1+u) - e^{-t}\int\left(1-\frac{1}{1+u}\right)du\right)$

$= \frac{1}{4}(e^{t}\log(1+u) - e^{-t}(u - \log(1+u))$

$= \frac{1}{4}(e^{t}\log(1+e^{2t}) - e^{-t}(e^{2t} - \log(1+e^{2t}))$

$= \frac{1}{4}\left(x\log(1+x^2) - \frac{1}{x}(x^2 - \log(1+x^2)\right)$

$= \frac{1}{4}\left(x+\frac{1}{x}\right)\log(1+x^2) - \frac{x}{4}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = \frac{c_1}{x} + c_2 x + \frac{1}{4}\left(x+\frac{1}{x}\right)\log(1+x^2) - \frac{x}{4}$

$\Rightarrow y = \frac{c_1}{x} + c_3 x + \frac{1}{4}\left(x+\frac{1}{x}\right)\log(1+x^2)$, $c_3 = c_2 - \frac{1}{4}$

Example 29 Solve the differential equation:

$x^2D^2 - 2x\text{D} - 4y = x^2 + 2\log x$, $x > 0$ ....①

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t$ $\therefore \log x = t$

$\Rightarrow xD = \delta y, x^2D^2 = \delta(\delta-1)y$, $\delta \equiv \frac{d}{dt}$

$\therefore$① May be rewritten as

$(\delta(\delta-1)-2\delta-4)y = e^{2t}+2t$

$\Rightarrow (\delta^2-3\delta-4)y = e^{2t}+2t$

Auxiliary equation is: $m^2-3m-4=0$

$\Rightarrow (m+1)(m-4)=0$

$\Rightarrow m=-1,4$

C.F. $= c_1e^{-t}+c_2e^{4t}$

$= \frac{c_1}{x}+\frac{c_2}{x^4}$

P.I. $= \frac{1}{f(\delta)}F(x) = \frac{1}{\delta^2-3\delta-4}(e^{2t}+2t)$

$= \frac{1}{\delta^2-3\delta-4}e^{2t}+\frac{1}{\delta^2-3\delta-4}2t$

$= \frac{1}{-6}e^{2t}+2\frac{1}{-4\left(1-\frac{\delta^2}{4}+\frac{3\delta}{4}\right)}t$ Putting $\delta=2$ in the 1st term

$= \frac{-e^{2t}}{6}-\frac{1}{2}\left(1-\left(\frac{\delta^2}{4}-\frac{3\delta}{4}\right)\right)^{-1}t$

$= \frac{-e^{2t}}{6}-\frac{1}{2}\left[1+\frac{\delta^2}{4}-\frac{3\delta}{4}+\cdots\right]t$

$= \frac{-e^{2t}}{6}-\frac{1}{2}\left[t-\frac{3}{4}\right]$

$\therefore$ P. I. $= \frac{-x^2}{6}-\frac{1}{2}\left[\log x-\frac{3}{4}\right]$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = \frac{c_1}{x}+\frac{c_2}{x^4}\,\frac{-x^2}{6}-\frac{1}{2}\left[\log x-\frac{3}{4}\right]$

3.9.2 Legendre's Linear Differential Equation

The differential equation of the form: $k_0(ax+b)^n\frac{d^ny}{dx^n}+$

$$k_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+k_{n-1}(ax+b)\frac{dy}{dx}+k_ny = F(x)$$

is called Legendre's linear equation and it can be reduced to linear differential equations with constant coefficients by following substitutions:

$(ax+b) = e^t \quad \Rightarrow t = \log(ax+b)$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{a}{ax+b}$

$\Rightarrow (ax+b)\frac{dy}{dx} = a\frac{dy}{dt} = aDy$, where $D \equiv \frac{d}{dt}$

Similarly$(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y$

$(ax+b)^3\frac{d^3y}{dx^3} = a^3D(D-1)(D-2)y$ and so on.

Example 30 Solve the differential equation:

$$(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1 \quad ....①$$

Solution: This is a Legendre's linear equation with variable coefficients.

Putting $(3x+2) = e^t \quad \therefore t = \log(3x+2)$

$\Rightarrow (3x+2)\frac{dy}{dx} = 3Dy$, $(3x+2)^2\frac{d^2y}{dx^2} = 3^2D(D-1)y$

Also $3x^2 + 4x + 1 = \frac{1}{3}(9x^2 + 12x + 3)$

$$= \frac{1}{3}((3x)^2 + 2.3.2x + 4 - 4 + 3)$$

$$= \frac{1}{3}((3x+2)^2 - 1)$$

$$= \frac{1}{3}(e^{2t} - 1)$$

$\therefore$① May be rewritten as

$(9D(D-1) + 9D - 36)y = \frac{1}{3}(e^{2t} - 1)$

$\Rightarrow 9(D^2 - 4)y = \frac{1}{3}(e^{2t} - 1)$

Auxiliary equation is: $9(m^2 - 4) = 0$

$\Rightarrow m = \pm 2$

C.F. $= c_1e^{-2t} + c_2e^{2t}$

$= \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{9(D^2-4)} \frac{1}{3}(e^{2t} - 1)$

$$= \frac{1}{27}\left(\frac{1}{(D^2-4)} e^{2t} - \frac{1}{(D^2-4)} e^{0t}\right)$$

$$= \frac{1}{27}\left(\frac{t}{2.2} e^{2t} - \frac{1}{(0-4)} e^{0t}\right)$$, Putting $D = 2$ in 1st term, it is a case of failure $\therefore \frac{1}{(D^2-4)} e^{2t} = t \frac{1}{f'(2)} e^{2x}$, also $D = 0$ in the 2nd term.

$\therefore$ P.I. $= \frac{1}{27}\left(\frac{t}{4} e^{2t} + \frac{1}{4}\right)$

$$= \frac{1}{27}\left(\frac{\log(3x+2)}{4} (3x+2)^2 + \frac{1}{4}\right)$$

$$= \frac{1}{108}[(3x+2)^2 \log(3x+2) + 1]$$

Complete solution is: $y =$ C.F. + P.I

$$\Rightarrow y = \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2 + \frac{1}{108}[(3x+2)^2 \log(3x+2) + 1]$$

Example 31 Solve the differential equation:

$$(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 2\sin(\log(x+1)), \quad x > -1 \text{ ....①}$$

Solution: This is a Legendre's linear equation with variable coefficients.

Putting $(x+1) = e^t$ $\therefore$ $t = \log(x+1)$

$$\Rightarrow (x+1)\frac{dy}{dx} = Dy, (x+1)^2 \frac{d^2y}{dx^2} = 1^2 D(D-1)y$$

∴① May be rewritten as

$$(D(D-1) + D + 1)y = 2\sin t$$

$$\Rightarrow (D^2 + 1)y = 2\sin t$$

Auxiliary equation is: $(m^2 + 1) = 0$

$$\Rightarrow m = \pm i$$

C.F. $= c_1 \cos t + c_2 \sin t$

$$= c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1))$$

$= \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+1} 2\sin t$

$= 2t\frac{1}{2D}\sin t$, Putting $D^2 = -1$, case of failure

$\therefore \frac{1}{(D^2+1)}\sin t = t\frac{1}{f'(D)}\sin t$

$= t\int \sin t\, dt = -t\cos t$

$\therefore$ P.I$= -\log(x+1)\cos(\log(x+1))$

Complete solution is: y = C.F. + P.I

$y = c_1\cos(\log(x+1)) + c_2\sin(\log(x+1)) - \log(x+1)\cos(\log(x+1))$

3.10 Method of Variation of Parameters for Finding Particular Integral

Method of Variation of Parameters enables us to find the solution of 2^{nd} and higher order differential equations with constant coefficients as well as variable coefficients.

Working rule

Consider a 2^{nd} order linear differential equation:

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = F(x)$ ....①

1. Find complimentary function given as: C.F. = $c_1y_1 + c_2y_2$,

   where y_1 and y_2 are two linearly independent solutions of ①

2. Calculate $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, W is called Wronskian of y_1 and y_2

3. Compute $u_1 = -\int \frac{y_2F(x)}{W}dx$, $u_2 = \int \frac{y_1F(x)}{W}dx$
4. Find P.I. = $u_1y_1 + u_2y_2$
5. Complete solution is given by: y = C.F. + P.I

Note: Method is commonly used to solve 2^{nd} order differential but it can be extended to solve differential equations of higher orders.

Example 32 Solve the differential equation: $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

using method of variation of parameters.

Solution: $\Rightarrow (D^2+1)y = \operatorname{cosec} x$

Auxiliary equation is: $(m^2+1) = 0$

$\Rightarrow\ m = \pm i$

C.F. = $c_1 \cos x + c_2 \sin x = c_1 y_1 + c_2 y_2$

$\therefore y_1 = \cos x$ and $y_2 = \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \sin x\ \operatorname{cosec} x\ dx = -\int 1 dx = -x$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \cos x\ \operatorname{cosec} x\ dx = \int \cot x\, dx = log|\sin x|$

$\therefore$ P.I= $u_1 y_1 + u_2 y_2$

$= -x \cos x + \sin x\ log|\sin x|$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x\ log|\sin x|$

Example 33 Solve the differential equation: $(D^2 - 2D + 1)y = e^x$

using method of variation of parameters.

Solution: Auxiliary equation is: $(m^2 - 2m + 1) = 0$

$\Rightarrow\ m = 1,1$

C.F. = $(c_1 + c_2\, x)e^x = c_1 e^x + c_2 x\, e^x = c_1 y_1 + c_2 y_2$

$\therefore y_1 = e^x$ and $y_2 = x\, e^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^x & x\, e^x \\ e^x & x\, e^x + e^x \end{vmatrix} = e^{2x}$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{xe^x e^x}{e^{2x}}\ dx = -\int x dx = -\frac{x^2}{2}$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^x e^x}{e^{2x}}\ dx = \int 1 dx = x$

$\therefore$ P.I$= u_1y_1 + u_2y_2$

$$= -\frac{x^2}{2}e^x + x^2e^x = \frac{x^2}{2}e^x$$

Complete solution is: y = C.F. + P.I

$$\Rightarrow y = (c_1 + c_2 x)e^x + \frac{x^2}{2}e^x$$

Example 34 Solve the differential equation: $\frac{d^2y}{dx^2} + 4y = x \sin 2x$

using method of variation of parameters.

Solution: $\Rightarrow (D^2 + 4)y = x \sin 2x$

Auxiliary equation is: $(m^2 + 4) = 0$

$\Rightarrow m = \pm 2i$

C.F. = $c_1 \cos 2x + c_2 \sin 2x = c_1y_1 + c_2y_2$

$\therefore y_1 = \cos 2x$ and $y_2 = \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$u_1 = -\int \frac{y_2F(x)}{W} dx = -\frac{1}{2}\int x \sin^2 2x \, dx = -\frac{1}{4}\int x(1 - \cos 4x)\, dx$$

$$= -\frac{1}{4}\left[\frac{x^2}{2} - \left[(x)\left(\frac{\sin 4x}{4}\right) - (1)\left(-\frac{\cos 4x}{16}\right)\right]\right]$$

$$= \left[-\frac{x^2}{8} + \frac{x \sin 4x}{16} + \frac{\cos 4x}{64}\right]$$

$$u_2 = \int \frac{y_1F(x)}{W} dx = \frac{1}{2}\int x \sin 2x \cos 2x \, dx = \frac{1}{4}\int x \sin 4x \, dx$$

$$= \frac{1}{4}\left[(x)\left(-\frac{\cos 4x}{4}\right) - (1)\left(-\frac{\sin 4x}{16}\right)\right]$$

$$= \left[-\frac{x \cos 4x}{16} + \frac{\sin 4x}{64}\right]$$

$\therefore$ P.I$= u_1y_1 + u_2y_2$

$$= \cos 2x\left[-\frac{x^2}{8} + \frac{x \sin 4x}{16} + \frac{\cos 4x}{64}\right] + \sin 2x\left[-\frac{x \cos 4x}{16} + \frac{\sin 4x}{64}\right]$$

$= \frac{x}{16}(\sin 4x \cos 2x - \cos 4x \sin 2x) + \frac{1}{64}(\cos 4x \cos 2x + \sin 4x \sin 2x)$

$-\frac{x^2}{8}\cos 2x \quad = \frac{x}{16}\sin 2x + \frac{1}{64}\cos 2x - \frac{x^2}{8}\cos 2x$

Complete solution is: $y = \text{C.F.} + \text{P.I}$

$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{16}\sin 2x + \frac{1}{64}\cos 2x - \frac{x^2}{8}\cos 2x$

Example 35 Solve the differential equation: $(D^2 - D - 2)y = e^{(e^x+3x)}$

using method of variation of parameters.

Solution: Auxiliary equation is: $(m^2 - m - 2) = 0$

$\Rightarrow m = -1,2$

C.F. $= c_1 e^{-x} + c_2 e^{2x} = c_1 y_1 + c_2 y_2$

$\therefore y_1 = e^{-x}$ and $y_2 = e^{2x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 3e^x$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{e^{2x} e^{(e^x+3x)}}{3e^x} dx = -\int \frac{e^{2x} e^{e^x} e^{3x}}{3e^x} dx$

$= -\frac{1}{3}\int e^{4x} e^{e^x} dx$, Putting $e^x = t \Rightarrow e^x dx = t dt$

$u_1 = -\frac{1}{3}\int t^3 e^t dt = -\frac{1}{3}[(t^3)(e^t) - (3t^2)(e^t) + (6t)(e^t) - (6)(e^t)]$

$\Rightarrow u_1 = -\frac{e^{e^x}}{3}[e^{3x} - 3e^{2x} + 6e^x - 6]$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^{-x} e^{(e^x+3x)}}{3e^x} dx = \int \frac{e^{-x} e^{e^x} e^{3x}}{3e^x} dx \; \frac{1}{3}\int e^x e^{e^x} dx = \frac{e^{e^x}}{3}$

$\therefore$ P.I$= u_1 y_1 + u_2 y_2$

$= -\frac{e^{e^x} e^{-x}}{3}[e^{3x} - 3e^{2x} + 6e^x - 6] + \frac{e^{e^x} e^{2x}}{3}$

$= \frac{e^{e^x}}{3}[3e^x - 6 + 6e^{-x}]$

Complete solution is: $y = \text{C.F.} + \text{P.I}$

$\Rightarrow y = c_1 e^{-x} + c_2\, e^{2x} + \frac{e^{e^x}}{3}[3e^x - 6 + 6e^{-x}]$

Example 36 Given that $\therefore y_1 = x$ and $y_2 = \frac{1}{x}$ are two linearly independent solutions of the differential equation: $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x, x \neq 0$

Find the particular integral and general solution using method of variation of parameters.

Solution: Rewriting the equation as: $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x}$

Given that $\therefore y_1 = x$ and $y_2 = \frac{1}{x}$

$\therefore$C.F. = $c_1 y_1 + c_2 y_2 = c_1 x + \frac{c_2}{x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{'} & y_2{'} \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{2}{x}$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = \int \frac{1}{x}.\frac{1}{x}.\frac{x}{2}\, dx = \frac{1}{2}\int \frac{1}{x} dx = \frac{1}{2}\log x$

$u_2 = \int \frac{y_1 F(x)}{W} dx = -\int x.\frac{1}{x}.\frac{x}{2}\, dx = -\frac{x^2}{4}$

$\therefore$ P.I= $u_1 y_1 + u_2 y_2$

$= \frac{x}{2}\log x - \frac{x}{4}$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1 x + \frac{c_2}{x} + \frac{x}{2}\log x - \frac{x}{4}$

Example 37 Solve the differential equation: $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^2 \log x$

using method of variation of parameters.

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy, x^2\frac{d^2y}{dx^2} = \mathrm{D(D-1)y}$

∴ Given differential equation may be rewritten as

$(D(D-1)-4D+6)y = te^{2t}$

$\Rightarrow (D^2-5D+6)y = te^{2t}$

Auxiliary equation is: $(m-2)(m-3)=0$

$\Rightarrow\ m = 2,3$

C.F. = $c_1e^{2t}+c_2e^{3t} = c_1y_1+c_2y_2$

$\therefore y_1 = e^{2t}$ and $y_2 = e^{3t}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t}$$

$u_1 = -\int \frac{y_2F(t)}{W}dt = -\int \frac{e^{3t}te^{2t}}{e^{5t}}\,dt = -\int t dt = -\frac{t^2}{2}$

$u_2 = \int \frac{y_1F(t)}{W}dt = \int \frac{e^{2t}te^{2t}}{e^{5t}}\,dt = \int te^{-t}dt = [(t)(-e^{-t})-(1)(e^{-t})]$

$= -te^{-t}-e^{-t}$

∴ P.I= $u_1y_1+u_2y_2$

$= -\frac{t^2}{2}e^{2t}-(te^{-t}+e^{-t})e^{3t}$

$= -\frac{t^2}{2}e^{2t}-te^{2t}-e^{2t} = -e^{2t}\left(\frac{t^2}{2}+t+1\right)$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1e^{2t}+c_2e^{3t}-e^{2t}\left(\frac{t^2}{2}+t+1\right)$

or $y = c_1x^2+c_2\,x^3-x^2\left(\frac{(\log x)^2}{2}+\log x+1\right)$

$\Rightarrow y = c_3x^2+c_2\,x^3-\frac{x^2}{2}(\log x)^2-x^2\log x,\ c_3 = c_1-1$

3.11 Solving Simultaneous Linear Differential Equations

Linear differential equations having two or more dependent variables with single independent variable are called simultaneous differential equations and can be of two types:

Type 1: $\boldsymbol{f_1(D)x + f_2(D)y = F(t)\,,\; g_1(D)x + g(D)y = G(t),\, D \equiv \frac{d}{dt}}$

Consider a system of ordinary differential equations in two dependent variables x and y and an independent variable t:

$$f_1(D)x + f_2(D)y = F(t)\,,\; g_1(D)x + g(D)y = G(t),\, D \equiv \frac{d}{dt}$$

Given system can be solved as follows:

1. Eliminate y from the given system of equations resulting a differential equation exclusively in x.
2. Solve the differential equation in x by usual methods to obtain x as a function of t .
3. Substitute value of x and its derivatives in one of the simultaneous equations to get an equation in y.
4. Solve for y by usual methods to obtain its value as a function of t.

Example 38 Solve the system of equations: $\frac{dx}{dt} + y = e^t,\; \frac{dy}{dt} - x = e^{-t}$

Solution: Rewriting given system of differential equations as:

$Dx + y = e^t$ ......①
$Dy - x = e^{-t}$ .....②, $D \equiv \frac{d}{dt}$
Multiplying ① by D
$\Rightarrow D^2x + Dy = e^t$ .....③
Subtracting ② from ③, we get
$(D^2 + 1)x = e^t - e^{-t}$ ....④

which is a linear differential equation in x with constant coefficients.

To solve ④ for x, Auxiliary equation is $m^2 + 1 = 0$

$\Rightarrow m = \pm i$

C.F. = $c_1 \cos t + c_2 \sin t$

P.I. = $\frac{1}{f(D)} F(t) = \frac{1}{D^2+1}(e^t - e^{-t}) = \frac{1}{D^2+1} e^t - \frac{1}{D^2+1} e^{-t}$

$= \frac{1}{2} e^t - \frac{1}{2} e^{-t}$, Putting $D = 1$ and $D = -1$ in 1st and 2nd terms respectively

$\therefore x = c_1 \cos t + c_2 \sin t + \frac{1}{2} e^t - \frac{1}{2} e^{-t}$.......⑤

Using ⑤in ① $\Rightarrow D\left[c_1 \cos t + c_2 \sin t + \frac{1}{2}e^t - \frac{1}{2}e^{-t}\right] + y = e^t$

$\Rightarrow \left[-c_1 \sin t + c_2 \cos t + \frac{1}{2}e^t + \frac{1}{2}e^{-t}\right] + y = e^t$

$\Rightarrow y = c_1 \sin t - c_2 \cos t + \frac{1}{2}e^t - \frac{1}{2}e^{-t}$ ......⑥

⑤ and ⑥ give the required solution.

Example 39 Solve the system of equations: $t\frac{dx}{dt} + y = 0, \ \frac{dy}{dt} + x = 0$

given that $x(1) = 1, \ y(-1) = 0$

Solution: Given system of equations is:

$t\frac{dx}{dt} + y = 0$......①

$t\frac{dy}{dt} + x = 0$ ......②,

Multiplying ① by $t\frac{d}{dt}$

$t\frac{d}{dt}(t\frac{dx}{dt} + y) = 0$

$\Rightarrow t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} + t\frac{dy}{dt} = 0$ ......③

Subtracting ② from ③, we get

$t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} - x = 0$.......④

which is Cauchy's linear differential equation in x with variable coefficients.

.Putting $t = e^k \quad \therefore \log t = k$

$\Rightarrow t\frac{dx}{dt} = Dx, t^2\frac{d^2x}{dt^2} = D(D-1)x \ , D \equiv \frac{d}{dk}$

$\therefore$ ④ may be rewritten as

$(D(D-1) + D - 1)x = 0$ ......⑤

$\Rightarrow (D^2 - 1)x = 0$

To solve ⑤ for x, Auxiliary equation is $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

C.F. = $c_1e^k + c_2e^{-k} = c_1t + \frac{c_2}{t}$

$\therefore x = c_1 t + \frac{c_2}{t}$.......⑥

Using ⑥in ① $\Rightarrow t\frac{d}{dt}\left(c_1 t + \frac{c_2}{t}\right) + y = 0$

$\Rightarrow c_1 t - \frac{c_2}{t} + y = 0$

$\Rightarrow y = -c_1 t + \frac{c_2}{t}$ ....⑦

Also given that at $t = 1, x = 1$ and at $t = -1, y = 0$

Using in ⑥and ⑦ $c_1 + c_2 = 1,\ \ c_1 - c_2 = 0 \Rightarrow c_1 = c_2 = \frac{1}{2}$

Using $c_1 = c_2 = \frac{1}{2}$ in ⑥and ⑦, we get

$x = \frac{1}{2}\left(t + \frac{1}{t}\right),\ y = \frac{1}{2}\left(\frac{1}{t} - t\right)$

Example 40 Solve the system of equations:

$\frac{d^2x}{dt^2} + y = \sin t,\ \ \frac{d^2y}{dt^2} + x = \cos t$

Solution: Rewriting given system of differential equations as:

$D^2x + y = \sin t$......①
$D^2y + x = \cos t$ .....②, $D \equiv \frac{d}{dt}$
Multiplying ① by D^2
$D^2(D^2x + y) = D^2\sin t$
$\Rightarrow D^4x + D^2y = -\sin t$ .....③
Subtracting ② from ③, we get
$(D^4 - 1)x = -\sin t - \cos t$ ....④

which is a linear differential equation in x with constant coefficients.

To solve ④ for x, Auxiliary equation is $m^4 - 1 = 0$

$\Rightarrow (m^2 - 1)(m^2 + 1) = 0$

$\Rightarrow m = \pm 1, \pm i$

C.F. $= c_1e^t + c_2e^{-t} + (c_3\cos t + c_4\sin t)$

P.I. $= \frac{1}{f(D)}F(t) = \frac{1}{D^4-1}(-\sin t - \cos t) = -\frac{1}{D^4-1}\sin t - \frac{1}{D^4-1}\cos t$

Putting $D^2 = -1$ i.e. $D^4 = 1$ in 1st and 2nd terms, it is a case of failure

$\therefore$ P.I. $= -t\frac{1}{4D^3}\sin t - t\frac{1}{4D^3}\cos t$

$= \frac{t}{4}\frac{1}{D}\sin t + \frac{t}{4}\frac{1}{D}\cos t$ putting $D^2 = -1$

$= -\frac{t}{4}\cos t + \frac{t}{4}\sin t$

$\therefore x = (c_1e^t + c_2e^{-t}) + (c_3\cos t + c_4\sin t) + \frac{t}{4}(\sin t - \cos t)$.......⑤

Using ⑤in ①

$\Rightarrow D^2\left[c_1e^t + c_2e^{-t} + (c_3\cos t + c_4\sin t) + \frac{t}{4}(\sin t - \cos t)\right] + y = \sin t$

$\Rightarrow D\left[c_1e^t - c_2e^{-t} - c_3\sin t + c_4\cos t + \frac{t}{4}(\cos t + \sin t) + \frac{1}{4}(\sin t - \cos t)\right]$
$+ y = \sin t$

$\Rightarrow \left[c_1e^t + c_2e^{-t} - c_3\cos t - c_4\sin t + \frac{t}{4}(-\sin t + \cos t) + \frac{1}{4}(\cos t + \sin t)\right.$
$\left. + \frac{1}{4}(\cos t + \sin t)\right] + y = \sin t$

$\Rightarrow y = -(c_1e^t + c_2e^{-t}) + (c_3\cos t + c_4\sin t) + \left(\frac{t}{4} + \frac{1}{2}\right)(\sin t - \cos t)$ ...⑥

⑤ and ⑥ give the required solution.

Type II: Symmetric simultaneous equations of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Simultaneous differential equations in the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ can be solved by the method of grouping or the method of multipliers or both to get two independent solutions: $u = c_1, v = c_2$; where c_1 and c_2 are arbitrary constants.

Method of grouping: In this method, we consider a pair of fractions at a time which can be solved for an independent solution.

Method of multipliers: In this method, we multiply each fraction by suitable multipliers (not necessarily constants) such that denominator becomes zero.

If a, b, c are multipliers, then $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{adx+bdy+cdz}{aP+bQ+cR}$

Example 41 Solve the set of simultaneous equations:

$$\frac{dx}{(z^2-2yz-y^2)} = \frac{dy}{(xy+zx)} = \frac{dz}{(xy-zx)}$$

Taking x, y, z as multipliers, each fraction equals

$$\frac{xdx+ydy+zdz}{(xz^2-2xyz-xy^2+xy^2+xyz+xyz-xz^2)} = \frac{xdx+ydy+zdz}{0}$$

$\Rightarrow\ xdx + ydy + zdz = 0$

Integrating, we get $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c'_1$

1st independent solution is: $u = x^2 + y^2 + z^2 = c_1$......①

Now for 2nd independent solution, taking last two members of the set of equations: $\frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$

$\Rightarrow (y-z)dy = (y+z)dz$

$\Rightarrow\ ydy - (zdy + ydz) - zdz = 0$

$\Rightarrow\ ydy - d(yz) - zdz = 0$

Integrating, we get

$\frac{y^2}{2} - yz - \frac{z^2}{2} = c'_2$

$\Rightarrow\ v = y^2 - 2yz - z^2 = c_2$ ....②

① and ② give the required solution.

Exercise 3.4

Q1. Solve the following differential equations:

i. $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=(1+x)^2$

Ans. $\langle y=(c_1+c_2\log x)x^2+\frac{1}{4}+2x+\frac{x^2}{2}(\log x)^2\rangle$

ii. $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=e^x$

Ans. $\langle y=\frac{c_1}{x}+\frac{c_2}{x^2}+\frac{e^x}{x^2}\rangle$

iii. $(2x+3)^2\frac{d^2y}{dx^2}-(2x+3)\frac{dy}{dx}-12y=6x$

Ans. $\langle y=c_1(2x+3)^{\frac{3+\sqrt{57}}{4}}+c_2(2x+3)^{\frac{3-\sqrt{57}}{4}}-\frac{3}{14}(2x+3)+\frac{3}{4}\rangle$

iv. $(x+1)^2\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}+y=4\cos(\log(x+1))$

Ans.$\langle y=c_1\cos(\log(x+1))+c_2\,sin(\log(x+1))+2\log(x+$
$1.sin\log x+1$

Q2. Solve the following differential equations using method of variation of parameters

i. $\frac{d^2y}{dx^2}+y=xsinx$

Ans. $\langle y=c_1\cos x+c_2\sin x+\frac{1}{8}\cos x+\frac{x}{4}\sin x-\frac{x^2}{4}\cos x\rangle$

ii. $(D^2-1)y=e^{-2x}\sin e^{-x}$

Ans. $\langle y=c_1e^x+c_2e^{-x}-\sin e^{-x}-e^x\cos e^{-x}\rangle$

iii. $(D^2-2D)y=e^x\sin x$

Ans. $\langle y=c_1+c_2e^{2x}-\frac{1}{2}e^x\sin x\rangle$

iv. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}=e^x\log x$

Ans: $\langle y=c_1+c_2e^{2x}+\frac{x^2}{4}e^x(2\log x-3)\rangle$

Q2. Solve the following set of simultaneous differential equations

i. $\frac{dx}{dt}-7x+y=0,\ \frac{dy}{dt}-2x-5y=0$

Ans:$\langle x=\ e^{6t}(c_1\cos t+c_2\sin t), y=e^{6t}(c_1-c_2)\cos t+(c_1+c_2)\sin t)\ \rangle$

ii. $(D+1)x+(2D+1)y=e^t\ ,(D-1)x+(D+1)y=1$

Ans: $\langle x = c_1e^{t} + c_2e^{-2t} + 2e^{-t}, y = 3c_1e^{t} + 2c_2e^{-2t} + 3e^{-t}\rangle$

iii. $\frac{dx}{(z^2-2yz-y^2)} = \frac{dy}{(xy+zx)} = \frac{dz}{(xy-zx)}$

Ans: $\langle xy - z = c_1\ ,\ x^2 - y^2 + z^2 = c_2\rangle$

Chapter 4
Partial Differential Equations

4.1 Introduction

A differential equation which involves partial derivatives is called partial differential equation (PDE). The order of a PDE is the order of highest partial derivative in the equation and the degree of PDE is the degree of highest order partial derivative occurring in the equation. Thus order and degree of the PDE $\left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^3 + \left(\frac{\partial z}{\partial x}\right)^4 - x\frac{\partial z}{\partial x} = 0$ are respectively 2 and 3.

If 'z' is a function of two independent variables 'x' and 'y', let us use the following notations for the partial derivatives of 'z' :

$$\frac{\partial z}{\partial x} \equiv p \ , \quad \frac{\partial z}{\partial y} \equiv q \ , \quad \frac{\partial^2 z}{\partial x^2} \equiv r, \quad \frac{\partial^2 z}{\partial x\, \partial y} \equiv s, \quad \frac{\partial^2 z}{\partial y^2} \equiv t$$

4.2 Linear Partial Differential Equations of 1st Order

If in a 1st order PDE, both 'p' and 'q' occur in 1st degree only and are not multiplied together, then it is called a linear PDE of 1st order, i.e. an equation of the form $Pp + Qq = R;\ P, Q, R$ are functions of x, y, z , is a linear PDE of 1st order.

Lagrange's Method to Solve a Linear PDE of 1st Order (Working Rule) :

4.2.1 Form the auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

4.2.2 Solve the auxiliary equations by the method of grouping or the method of multipliers* or both to get two independent solutions: $u = a, v = b$; where a and b are arbitrary constants.

4.2.3 $\varphi\,(u, v) = 0$ or $u = f(v)$ is the general solution of the equation $Pp + Qq = R$.

*__Method of multipliers__: Consider a fraction $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$

Taking 1,2, 3 as multipliers, each fraction $= \frac{1\times1}{2\times1} = \frac{2\times2}{4\times2} = \frac{3\times3}{6\times3}$

Example 1. Solve the PDE $(z^2-2yz-y^2)p+(xy+zx)q=xy-zx$

Solution: Comparing with general form $P\equiv(z^2-2yz-y^2), Q\equiv(xy+zx)$,

$R\equiv(xy-zx)$

Step 1.

Auxiliary equations are $\frac{dx}{(z^2-2yz-y^2)}=\frac{dy}{(xy+zx)}=\frac{dz}{(xy-zx)}$

Step 2.

Taking x,y,z as multipliers, each fraction $=\frac{xdx+ydy+zdz}{(xz^2-2xyz-xy^2+xy^2+xyz+xyz-xz^2)}$

$$=\frac{xdx+ydy+zdz}{0}$$

$\Rightarrow\ xdx+ydy+zdz=0$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=a_1$$

$\Rightarrow u=\ x^2+y^2+z^2=a$ ________ ①

This is 1st independent solution.

Now for 2nd independent solution, taking last two members of auxiliary equations:

$$\frac{dy}{x(y+z)}=\frac{dz}{x(y-z)}$$

$\Rightarrow(y-z)dy=(y+z)dz$

$\Rightarrow\ ydy-(zdy+ydz)-\ zdz=0$

$\Rightarrow\ ydy-d(yz)-\ zdz=0$

Integrating, we get

$$\frac{y^2}{2}-yz-\frac{z^2}{2}=b_1$$

$\Rightarrow\ \ v=\ y^2-2yz-z^2=b$ ________ ②

Which is 2nd independent solution

From ① and ②, general solution is:

$$\varphi\,(x^2+y^2+z^2,\ y^2-2yz-z^2)=\ 0$$

4.3 Homogenous Linear Equations with Constant Coefficients

An equation of the form

$$k_0\,\frac{\partial^n z}{\partial x^n}+\ k_1\,\frac{\partial^n z}{\partial x^{n-1}\partial y}+-----+k_n\,\frac{\partial^n z}{\partial y^n}=F(x,y) \quad \text{③}$$

where $k's$ are constant is called a homogeneous linear PDE of n^{th} order with constant coefficients. It is homogeneous because all the terms contain derivatives of the same order.

③ may be written as:

$$(k_0\,D^n+\ k_1\,D^{n-1}D'+-----+k_n\,D'^n)z=F(x,y)$$

or $f(\,D,D')$ z = $F(x,y)$, where $\frac{\partial}{\partial x}\equiv D$ and $\frac{\partial}{\partial y}\equiv D'$

4.3.1 Solving Homogenous Linear Equations with Constant Coefficients

Case 1: When $F(x,y)=0$

i.e. equation is of the form $k_0\,\frac{\partial^2 z}{\partial x^2}+\ k_1\,\frac{\partial^2 z}{\partial x\partial y}+\ k_2\,\frac{\partial^2 z}{\partial y^2}=0$--------- ④

or $k_0\,D^2+\ k_1\,DD'+\ k_2\,D'^2=0$

In this case Z = C.F.

Case 2: When $F(x,y)\ \neq 0$

i.e. equation is of the form $k_0\,\frac{\partial^2 z}{\partial x^2}+\ k_1\,\frac{\partial^2 z}{\partial x\partial y}+\ k_2\,\frac{\partial^2 z}{\partial y^2}=F(x,y)$ -------- ⑤

or $k_0\,D^2+\ k_1\,DD'+\ k_2\,D'^2=F(x,y)$

In this case Z = C.F. + P.I.

Where C.F. denotes complimentary function and P.I. denotes Particular Integral.

Rules for finding C.F. (Complimentary Function)

Step 1: Put $D=m$ and $D'=1$ in ④or ⑤as the case may be Then A.E. (Auxiliary Equation) is: $k_0\,m^2+\ k_1\,m+\ k_2\ =0$

Step 2: Solve the A.E. (Auxiliary Equation):

i. If the roots of A.E. are real and different say m_1 and m_2, then

C.F. = $f_1(y+m_1x)+f_2(y+m_2x)$

ii. If the roots of A.E. are equal say m, then

C.F. = $f_1(y+mx)+x\,f_2(y+mx)$

Example 1.2 Solve $\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x\partial y}-5\frac{\partial^2 z}{\partial y^2}=0$

Solution:⇒ $(D^2-4DD'-5D'^2)z=0$

Auxiliary equation is: $m^2-4m-5=0$

$\Rightarrow (m-5)(m+1)=0$

$\Rightarrow m=5,-1$

C.F. = $f_1(y+5x)+f_2(y-x)$

⇒ Z = $f_1(y+5x)+f_2(y-x)$

Example 1.3 Solve $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0$

Solution:⇒ $(D^2+2DD'+D'^2)z=0$

Auxiliary equation is: $m^2+2m+1=0$

$\Rightarrow (m+1)^2=0$

$\Rightarrow m=-1,-1$

C.F. = $f_1(y-x)+x\,f_2(y-x)$

⇒z= $f_1(y-x)+x\,f_2(y-x)$

Rules for finding P.I. (Particular Integral)

* Applicable only if $F(x,y) \neq 0$

Let the given PDE be $f(D,D')$ z = $F(x,y)$

P.I = $\frac{F(x,y)}{f(D,D')}$

Case I : When $F(x,y) = e^{ax+by}$

Put $D = a$ and $D' = b$

If $f(a,b) \neq 0$, P.I = $\frac{e^{ax+by}}{f(a,b)}$

and if $f(a,b) = 0$, P.I = $\frac{x\, e^{ax+by}}{\frac{d}{dD}f(D,D')}$

Now put $D = a$ and $D' = b$

Example 1.4 Solve $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y}$

Solution:⇒ $(D^2 - 5DD' + 6D'^2)z = e^{x+y}$

Auxiliary equation is: $m^2 - 5m + 6 = 0$

$\Rightarrow (m-2)(m-3) = 0$

$\Rightarrow m = 2,3$

C.F. = $f_1(y + 2x) + f_2(y + 3x)$

P.I = $\frac{e^{x+y}}{D^2 - 5DD' + 6D'^2}$

Put $D = 1$, $D' = 1$

P.I = $\frac{e^{x+y}}{1-5+6} = \frac{e^{x+y}}{2}$

$\Rightarrow$ Z = $f_1(y + 2x) + f_2(y + 3x) + \frac{e^{x+y}}{2}$

Example 1.5 Solve $r - 4s + 4t = e^{2x+y}$

Solution:$\Rightarrow \frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$

$$\Rightarrow (D^2 - 4DD' + 4D'^2)z = e^{2x+y}$$

Auxiliary equation is: $m^2 - 4m + 4 = 0$

$\Rightarrow (m - 2)^2 = 0$

$\Rightarrow m = 2,2$

C.F. = $f_1(y + 2x) + xf_2(y + 2x)$

P.I = $\frac{e^{2x+y}}{(D-2D')^2}$

Putting $D = 2$ $D' = 1$, denominator = 0

P.I = $\frac{xe^{2x+y}}{\frac{d}{dD}(D-2D')^2} = \frac{xe^{2x+y}}{2(D-2D')}$

Putting $D = 2$, $D' = 1$, again denominator = 0

P.I. = $\frac{x^2 e^{2x+y}}{\frac{d}{dD}2(D-2D')}$

$\Rightarrow$ P.I. $= \dfrac{x^2 e^{2x+y}}{2}$

Complete solution is Z= C.F. + P.I.

$\Rightarrow$ Z= $f_1(y + 2x) + xf_2(y + 2x) + \dfrac{x^2 e^{2x+y}}{2}$

Case II : When $F(x,y) = e^{ax+by}\phi(x,y)$, $\phi(x,y)$ is a trigonometric function of sine or cosine.

$$\text{P.I} = e^{ax+by}\frac{1}{f(D+a,\ D'+b)}\phi(x,y)$$

Example 1.6 Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \sin 2y$

Solution: Auxiliary equation is: $m^3 + m^2 - m - 1 = 0$

$\Rightarrow m = 1, -1, -1$

C.F. = $f_1(y + x) + f_2(y - x) + xf_3(y - x)$

$$\text{P.I} = \frac{e^x \sin 2y}{D^3 + D^2D' - DD'^2 - D'^3}$$

$$= \frac{e^x \sin 2y}{D^2(D+D') - D'^2(D+D')} = \frac{e^x \sin 2y}{(D+D')^2(D-D')}$$

$$= e^x \frac{1}{f(D+1,\ D'\ )}\sin 2y \quad (\because\ a = 1, b = 0)$$

$$= e^x \frac{1}{(D+1+D')^2(D+1-D')}\sin 2y$$

$$= e^x \frac{1}{(D^2+1+D'^2+2D+2D'+2DD')(D+1-D')}\sin 2y$$

Put $D^2 = 0,\ DD' = 0\ ,\ D'^2 = -4$

$$\text{P.I.} = e^x \frac{1}{(0+1-4+2D+2D'+0)(D+1-D')} \sin 2y$$

$$= e^x \frac{1}{(-3+2D+2D')(D+1-D')} \sin 2y$$

$$= e^x \frac{1}{(-3D+2D^2+2DD'-3+2D+2D'+3D'-2DD'-2D'^2)} \sin 2y$$

$$= e^x \frac{1}{(-D+0+0-3+5D'-0+8)} \sin 2y$$

$$= e^x \frac{1}{(5D'-D)+5} \sin 2y$$

$$= e^x \frac{(5D'-D)-5}{(5D'-D)^2-25} \sin 2y$$

$$= e^x \frac{(5D'-D-5)\sin 2y}{25D'^2+D^2-10DD'-25}$$

$$= e^x \frac{(10\cos 2y-0-5\sin 2y)}{25(-4)+0+0-25}$$

$$= \frac{e^x}{-125}(10\cos 2y - 5\sin 2y)$$

$$= \frac{e^x}{25}(\sin 2y - 2\cos 2y)$$

$$\Rightarrow Z = f_1(y+x) + f_2(y-x) + x f_3(y-x) + \frac{e^x}{25}(\sin 2y - 2\cos 2y)$$

Case III : When $F(x,y) = Sin\,(ax+by)$ or $Cos\,(ax+by)$

P.I = $\frac{Sin(ax+by) \text{ or } Cos\,(ax+by)}{f(D^2, DD', D'^2)}$

Put $D^2 = -a^2,\ D' = -ab\ , D'^2 = -b^2$

Hence P.I = $\frac{Sin(ax+by) or Cos\,(ax+by)}{f(-a^2, -ab, -b^2)}$

Example 1.7 Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = Sinx\ Cos2y$

Solution:⇒ $(D^2 - DD')z = Sinx\ Cos2y$

Auxiliary equation is: $m^2 - m = 0$

$\Rightarrow m(m-1) = 0$

$\Rightarrow m = 0, 1$

C.F. = $f_1(y) + f_2(y+x)$

P.I = $\frac{Sinx\ Cos2y}{D^2 - DD'} = \frac{1}{D^2 - DD'} \frac{1}{2} [Sin\,(x+2y) + Sin(x-2y)]$

$$= \frac{1}{2}\frac{1}{D^2 - DD'} Sin(x+2y) + \frac{1}{2}\frac{1}{D^2 - DD'} Sin(x-2y)$$

Putting $D^2 = -1,\ DD' = -2$ in the 1st term, $D^2 = -1,\ DD' = 2$ in the 2nd term

$$= \frac{1}{2}\frac{Sin\,(x+2y)}{-1-(-2)} + \frac{1}{2}\frac{Sin\,(x-2y)}{-1-(2)}$$

$\Rightarrow$ P.I. $= \frac{1}{2}\, Sin(x+2y) - \frac{1}{6} Sin\,(x-2y)$

Complete solution is Z= C.F +P.I

$\Rightarrow$ Z= $f_1(y) + f_2(y+x) + \frac{1}{2}\, Sin(x+2y) - \frac{1}{6} Sin\,(x-2y)$

Case IV : When $F(x,y) = x^m y^n$

$$\text{P.I} = \frac{x^m y^n}{f(D,D')}$$

$$= [f(D,D')]^{-1}.x^m y^n$$

Expand $[f(D,D')]^{-1}$ in ascending powers of D or D' and operate on $x^m y^n$ term by term.

Example 1.8 Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = x + y$

Solution:

$$\Rightarrow \left(D^2 + DD' - 6D'^2\right)z = x + y$$

Auxiliary equation is: $m^2 + m - 6 = 0$

$$\Rightarrow (m+3)(m-2) = 0$$

$$\Rightarrow m = -3, 2$$

C.F. = $f_1(y - 3x) + f_2(y + 2x)$

$$\text{P.I} = \frac{x+y}{D^2 + DD' - 6D'^2}$$

$$= \frac{1}{D^2}\left[1 + \frac{D'}{D} - 6\frac{D'^2}{D^2}\right]^{-1}(x+y)$$

$$= \frac{1}{D^2}\left[1 - \frac{D'}{D} + - - - - -\right](x+y)$$

$$\because (1+t)^{-1} = 1 - t + t^2 - t^3 + - - -$$

$$= \frac{1}{D^2}\left[(x+y) - \frac{1}{D}(0+1)\right]$$

$= \frac{1}{D^2}[x + y - x]$

P.I. $= \frac{1}{D^2}[y] = \frac{yx^2}{2}$

Complete solution is Z= C.F +P.I

$\Rightarrow$ Z = $f_1(y - 3x) + f_2(y + 2x) + \frac{yx^2}{2}$

Example 1.9 Solve $(D^3 - 3\,D^2D')z = x^2y$

Solution: Auxiliary equation is: $m^3 - 3\,m^2 = 0$

$\Rightarrow m^2(m - 3) = 0$

$\Rightarrow\ m = 0,0,3$

C.F. = $f_1(y) + xf_2(y) + f_3(y + 3x)$

P.I = $\frac{x^2y}{D^3 - 3\,D^2D'}$

$= \frac{1}{D^3}\left[1 - \frac{3D'}{D}\right]^{-1}(x^2y)$

$= \frac{1}{D^3}\left[1 + \frac{3D'}{D}\right](x^2y)$

$\because (1 - t)^{-1} = 1 + t + t^2 + t^3 + - - -$

$= \frac{1}{D^3}\left[(x^2y) + \frac{3}{D}(x^2)\right]$

$= \frac{1}{D^3}[x^2y + x^3]$

$$=\frac{1}{D^2}\left[\frac{x^3 y}{3}+\frac{x^4}{4}\right]$$

$$=\frac{1}{D}\left[\frac{x^4 y}{12}+\frac{x^5}{20}\right]$$

$$\Rightarrow \text{P.I.} =\left[\frac{x^5 y}{60}+\frac{x^6}{120}\right]$$

Complete solution is Z= C.F +P.I

$$\Rightarrow \text{Z} = f_1(y)+ \; x f_2(y)+f_3(y+3x)+\left[\frac{x^5 y}{60}+\frac{x^6}{120}\right]$$

Case V: In case of any function of $F(x,y)$ or when solution fails for any case by above given methods

$$\text{P.I} = \frac{F(x,y)}{f(D,D')}$$

Resolve $\frac{1}{f(D,D')}$ into partial fractions considering $f(D,D')$ as function of D alone.

$$\text{P.I} = \frac{F(x,y)}{D-m\,D'} = \int F(x,c-mx)dx$$

where C is replaced by $y+mx$ after integration.

Example 1.10 Solve $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

Solution: Auxiliary equation is: $m^2 - m - 2 = 0$

$$\Rightarrow (m-2)(m+1)=0$$

$$\Rightarrow m = 2, -1$$

C.F. = $f_1(y+2x)+ \; f_2(y-x)$

P.I = $\frac{(y-1)e^x}{D^2 - DD' - 2D'^2} = \frac{(y-1)e^x}{D^2 - 2DD' + DD' - 2D'^2} = \frac{(y-1)e^x}{(D-2D')(D+D')}$

$$= \frac{1}{(D-2D')} \int (c + x - 1)\, e^x dx$$

Putting $y = c + x \;\; as \;\; m = -1$

$$= \frac{1}{(D-2D')} [(c + x - 1)e^x - e^x]$$

$$= \frac{1}{(D-2D')} [(c + x)e^x - 2e^x]$$

$$= \frac{1}{(D-2D')} [(y - x + x)e^x - 2e^x]$$

Putting $c = y - x$

$$= \frac{1}{(D-2D')} [ye^x - 2e^x]$$

$$= \frac{1}{(D-2D')} [(y - 2)e^x]$$

$$= \int (c - 2x - 2)e^x\, dx$$

Putting $y = c - 2x$

$$= (c - 2x - 2)e^x - (-2)e^x$$

$$= (c - 2x)e^x$$

$$= (y + 2x - 2x)e^x$$

Putting $c = y + 2x$

$\Rightarrow$P.I. = ye^{x}

Complete solution is Z= C.F +P.I

$\Rightarrow$ Z =$f_1(y+2x) + \; f_2(y-x) + \; ye^{x}$

Example 1.11 Solve $2\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 5sin(2x+y)$

Solution: $\Rightarrow \left(2D^2 - 5DD' + 2D'^2\right)z = 5sin(2x+y)$

Auxiliary equation is:$2m^2 - 5m + 2 = 0$

$\Rightarrow (2m-1)(m-2) = 0$

$\Rightarrow \; m = \frac{1}{2}, 2$

C.F. = $f_1\left(y + \frac{x}{2}\right) + \; f_2(y+2x)$

P.I = $\frac{5sin(2x+y)}{2D^2 - 5DD' + 2D'^2}$

Putting$D^2 = -4, \; DD' = -2, D'^2 = -1$,denominator =0

$\therefore$ solution fails as per case II, resolving denominator into partial fractions

P.I = $\frac{5sin(2x+y)}{(2D-D')(D-2D')}$

= $\frac{5}{(2D-D')}\int sin(2x + (c-2x))\,dx$

Putting $y = c - 2x$

$$= \frac{5}{2} \frac{1}{\left(D - \frac{D'}{2}\right)} \int sin\ c dx$$

$$= \frac{5}{2} \frac{x\ sinc}{\left(D - \frac{D'}{2}\right)}$$

$$= \frac{5}{2} \frac{x\ sin\,(y+2x)}{\left(D - \frac{D'}{2}\right)}$$

$$= \frac{5}{2} \int x\ sin\left[\left(c - \frac{x}{2}\right) + 2x\right] dx$$

Putting $y = c - \frac{x}{2}$

$$= \frac{5}{2} \int x\ sin(c + \frac{3}{2}x)\ dx$$

$$= \frac{5}{2}\left[(x)\frac{(-cos(c+\frac{3}{2}x)}{\frac{3}{2}} - (1)\frac{(-sin(c+\frac{3}{2}x)}{\frac{9}{4}}\right]$$

$$= -\frac{5}{3}x\ cos\left(c + \frac{3}{2}x\right) + \frac{10}{9} sin(c + \frac{3}{2}x)$$

$$= -\frac{5}{3}x\ cos\left(y + \frac{x}{2} + \frac{3}{2}x\right) + \frac{10}{9} sin(y + \frac{x}{2} + \frac{3}{2}x)$$

Putting $c = y + \frac{x}{2}$

$\Rightarrow$P.I. $= -\frac{5}{3}x\ cos(y + 2x) + \frac{10}{9} sin(y + 2x)$

Complete solution is Z= C.F +P.I

$\Rightarrow$ Z $= f_1\left(y + \frac{x}{2}\right) + f_2(y + 2x) - \frac{5}{3}x\ cos(y + 2x) + \frac{10}{9} sin(y + 2x)$

4.3.2 Non-Homogeneous Linear Equations

If in the equation $(D,D')z = F(x,y)$, the polynomial $f(D,D')$ in D, D' is not homogeneous, then it is called a non-homogeneous partial differential equation.

Working Rule to Solve a Non-Homogeneous-Homogeneous Linear Equation

Step1: Resolve $f(D,D')$ into linear factors of the form

$(D - m_1D' - a_1)(D - m_2D' - a_2)\ldots\ldots\ldots(D - m_nD' - a_n)$

Step2: Auxiliary equation is

$(D - m_1D' - a_1)(D - m_2D' - a_2)\ldots\ldots\ldots(D - m_nD' - a_n) = 0$

Step3: C.F.=$e^{a_1x}f_1(y + m_1x) + e^{a_2x}f_2(y + m_2x) + \cdots \ldots\ldots\ldots\ldots + e^{a_nx}f_n(y + m_nx)$

In case of two repeated factors

C.F.=$e^{ax}f_1(y + mx) + xe^{ax}f_2(y + mx)$

Step4: Find P.I. by using usual methods of homogeneous PDE.

Step5: Complete solution is Z = C.F. + P.I.

Note: If the Auxiliary equation is of the form

$(D' - m_1D - a_1)(D' - m_2D - a_2)\ldots\ldots\ldots(D' - m_nD - a_n) = 0$

Then C.F.=$e^{a_1y}f_1(x + m_1y) + e^{a_2y}f_2(x + m_2y) + \cdots \ldots\ldots\ldots\ldots + e^{a_ny}f_n(x + m_ny)$

Example 1.12 Solve $(D^2 - D'^2 - 3D + 3D')z = e^{x+2y}$

Solution: Auxiliary equation is:$(D^2 - D'^2 - 3D + 3D') = 0$

Clearly $D = D'$is satisfying the equation, $\therefore (D - D')$ is a factor.

Dividing by$(D - D')$, we get

$(D - D')(D + D' - 3) = 0$

$\Rightarrow (D - D' - 0)(D + D' - 3) = 0$

$\Rightarrow$ C.F.= $f_1(y+x)+ e^{3x} f_2(y-x)$

$$\text{P.I} = \frac{e^{x+2y}}{(D^2-D'^2-3D+3D')}$$

Putting $D = 1,\ D' = 2, f(a,b) = 0$

$$\Rightarrow \text{P.I} = \frac{xe^{x+2y}}{\frac{d}{dD}(D^2-D'^2-3D+3D')} = \frac{xe^{x+2y}}{2D-3}$$

Putting D = 1

P.I. = $-xe^{x+2y}$

Complete solution is Z= C.F. + P.I.

$\Rightarrow$ Z= $f_1(y+x)+ e^{3x} f_2(y-x) - xe^{x+2y}$

Example 1.13 Solve $(2DD' + D'^2 - 3D')z = 3\cos(3x-2y)$

Solution: Auxiliary equation is: $(2DD' + D'^2 - 3D') = 0$

$$.\Rightarrow D'(D' + 2D - 3) = 0$$

$\Rightarrow$ C.F.= $f_1(x)+ e^{3y} f_2(x-2y)$

$$\text{P.I} = \frac{3\cos(3x-2y)}{(2DD'+D'^2-3D')}$$

Putting $DD' = 6\ , D'^2 = -4$

$$\Rightarrow \text{P.I} = \frac{3\cos(3x-2y)}{(12-4-3D')} = \frac{3\cos(3x-2y)}{(8-3D')}$$

$$= \frac{3\,(8+3D')\cos(3x-2y)}{(8-3D')(8+3D')}$$

$$= \frac{3\,(8+3D')\cos(3x-2y)}{(64-9D'^2)}$$

$$= \frac{3\,(8+3D')\cos(3x-2y)}{(64+36)}$$

$$= \frac{3}{100}(8+3D')\cos(3x-2y)$$

$$= \frac{3}{100}[8\cos(3x-2y)+6\sin(3x-2y)]$$

$$= \frac{3}{50}[4\cos(3x-2y)+3\sin(3x-2y)]$$

Complete solution is Z= C.F. + P.I.

$$\Rightarrow Z = f_1(x) + e^{3y} f_2(x-2y) + \frac{3}{50}[4\cos(3x-2y)+3\sin(3x-2y)]$$

4.4 Applications of PDEs (Partial Differential Equations)

In this Section we shall discuss some of the most important PDEs that arise in various branches of science and engineering. Method of separation of variables is the most important tool, we will be using to solve basic PDEs that involve wave equation, heat flow equation and laplace equation.

Wave equation (vibrating string) : $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

One- dimensional heat flow (in a rod) : $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Two- dimensional heat flow in steady state (in a rectangular plate) :$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Note: Two-dimension heat flow equation in steady state is also known as Laplace equation.

Working Rule for Method of Separation of Variables

Let u be a function of two independent variables x and t

Step1. Assume the solution to be the product of two functions each of which involves only one variable.

Step 2. Calculate the respective partial derivative and substitute in the given PDE.

Step 3. Arrange the equation in the variable separable form and put LHS = RHS = K

(as both x and t are independent variables)

Step 4. Solve these two ordinary differential equations to find the two functions of x and t alone.

Example 1.14 Solve the equation $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial t} + u$ given that $u(x,0) = 5e^{-2x}$

Solution: Step1.

Let $u = XT$ .... ①

where X is a function of x alone and T be a function of t alone.

Step 2.

$\frac{\partial u}{\partial x} = X'T$, $\frac{\partial u}{\partial t} = XT'$

Substituting these values of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}$ in the given equation

$X'T = 4XT' + XT \quad \Rightarrow X'T = X(4T' + T)$

$Step$ 3.

$$\Rightarrow \frac{X'}{X} = \frac{4T'}{T} + 1$$

Putting LHS = RHS = K

Step 4.

i.e. $\frac{X'}{X} = K$

$\Rightarrow \log X = Kx + \log C_1$

$\Rightarrow \log \frac{X}{C_1} = Kx$

$\Rightarrow X = C_1 e^{Kx}$ ……. ②

$\frac{4T'}{T} + 1 = K$

$\frac{T'}{T} = \frac{1}{4}(K-1)$

$\Rightarrow \log T = \frac{1}{4}(K-1)t + \log C_2$

$\Rightarrow T = C_2 e^{\frac{1}{4}(K-1)t}$ ……… ③

Using ② and ③in ①

$u(x,t) = C_1 C_2 e^{Kx} e^{\frac{1}{4}(K-1)t}$ ……….④

$\Rightarrow u(x,0) = C_1 C_2 e^{Kx}$ ………. ⑤

Given that $u(x,0) = 5e^{-2x}$, using in ⑤

$\Rightarrow 5e^{-2x} = C_1 C_2 e^{Kx}$

$\Rightarrow C_1 C_2 = 5,\ K = -2$ …………….⑥

Using ⑥in ④

$u(x,t) = 5e^{-\left(2x+\frac{3t}{4}\right)}$

4.4.1 Solution of wave equation using method of separation of variables

Wave equation is given by $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ ……………… ①

where u gives displacement at distance x from origin at any time t. To solve wave equation using method of separation of variables,

Let $u = XT$ ………………….. ②

where X is a function of x alone and T be a function of t alone.

$\therefore \frac{\partial^2 u}{\partial t^2} = XT''$, $\frac{\partial^2 u}{\partial x^2} = X''T$

Substituting these values of $\frac{\partial^2 u}{\partial t^2}$, $\frac{\partial^2 u}{\partial x^2}$ in the wave equation given by ①

$XT'' = c^2 X'' T$

Arranging in variable separable form$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$

Equating LHS = RHS = K $(\because X \text{ and } T \text{ are independent})$

$\Rightarrow \frac{X''}{X} = K$ and $\frac{1}{c^2}\frac{T''}{T} = K$

$\Rightarrow X'' - KX = 0$ and $T'' - KC^2 T = 0$ ……………③

Solving ordinary differential equations given in ③, three cases arise

4.4.1.1 K is +ve and = p^2say

$$X = c_1 e^{px} + c_2 e^{-px}, T = c_3 e^{cpt} + c_4 e^{-cpt}$$

4.4.1.2 K is -ve and = $-p^2$say

$$X = c_1 cospx + c_2 sinpx, T = c_3 coscpt + c_4 sincpt$$

(iii)When K = 0

$$X = c_1 x + c_2, T = c_3 t + c_4$$

Again since we are dealing with wave equation, u must be a periodic function of $x$and t∴ the solution must involve trigonometric terms. Hence the solution given by (ii) i.e. corresponding to $K = -p^2$, is the most plausible solution, substituting (ii) in equation ② $u(x,t) = (c_1 cospx + c_2 sinpx)(c_3 coscpt + c_4 sincpt)$ …….. ④

Which is the required solution of wave equation.

Again if we consider a string of length l tied at both ends at $x = 0$ and $x = l$,

then displacement of the string at end points at any time t is zero .

$\Rightarrow u(0,t) = 0$ ……….. ⑤

and$u(l,t) = 0$ ………..⑥

using ⑤in ④$\Rightarrow 0 = C_1(c_3 coscpt + c_4 sincpt)$

$\Rightarrow C_1 = 0$ ……………. ⑦

Using ⑦ in ④ , wave equation reduces to

$u(x,t) = c_2 sinpx (c_3 coscpt + c_4 sincpt)$ …….. ⑧

Now using ⑥in ⑧⇒ 0 = $c_2 sinpl\,(c_3 coscpt + c_4 sincpt)$

⇒$\sin pl = 0 \quad \because c_2 \neq 0$ and $(c_3 coscpt + c_4 sincpt) \neq 0$

⇒$pl = n\pi$⇒$p = \frac{n\pi}{l}$, $n = 1,2,3 \ldots \ldots \ldots$---------⑨

using ⑨in ⑧⇒ $u(x,t) = c_2 sin\frac{n\pi x}{l}\,(c_3 \cos\frac{n\pi ct}{l} + c_4 sin\frac{n\pi ct}{l})$

Adding up the solutions for different values of n, we get

$u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{l} + b_n sin\frac{n\pi ct}{l}\right) sin\frac{n\pi x}{l}$.........⑩

⑩ is also a solution of wave equation

Example 1.15 : A string is stretched and fastened to 2 points l apart. Motion is started by displacing the string in the form $y = a\sin\frac{\pi x}{l}$ from which , it is released at time $t = 0$. Show that the displacement at any point at a distance x from one end at time t is given by

$$y(x,t) = a\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$$

Solution: Let the equation of vibrating string be given by $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ ...... ①

Boundary value conditions are given by :

$y(0,t) = 0$ ....②

$y(l,t) = 0$ ....③

$\frac{\partial y}{\partial t}_{t=0} = 0$............④

$y(x,0) = a\sin\frac{\pi x}{l}$......⑤

Let the solution of ①be given by

$y(x,t) = (c_1 cospx + c_2 sinpx)\,(c_3 coscpt + c_4 sincpt)$......⑥

Using ② in ⑥

⇒ 0 = $C_1(c_3 coscpt + c_4 sincpt)$

⇒ $C_1 = 0$ .... ⑦

Using ⑦ in ⑥ , wave equation reduces to

$y(x,t) = c_2 sinpx\,(c_3 coscpt + c_4 sincpt)$ …….. ⑧

Now using ③in⑧⇒ $0 = c_2 sinpl\,(c_3 coscpt + c_4 sincpt)$

$\Rightarrow \sin pl = 0 \quad \because c_2 \neq 0$ and $(\,c_3 coscpt + c_4 sincpt) \neq 0$

$\Rightarrow pl = n\pi \Rightarrow p = \frac{n\pi}{l}$ -------- ⑨

using ⑨in ⑧⇒ $y(x,t) = c_2 sin\frac{n\pi x}{l}\,(c_3 \cos\frac{n\pi ct}{l} + c_4 sin\frac{n\pi ct}{l})$……..⑩

Now to use ④, differentiating ⑩partially w.r.tt

$\Rightarrow \frac{\partial y}{\partial t} = c_2 sin\frac{n\pi x}{l}\,(-c_3\frac{n\pi c}{l}\sin\frac{n\pi ct}{l} + c_4\frac{n\pi c}{l} cos\frac{n\pi ct}{l})$

Putting $t = 0$

$\frac{\partial y}{\partial t}_{t=0} = c_2 sin\frac{n\pi x}{l}\left(c_4\frac{n\pi c}{l}\right) = 0$ using④

$\Rightarrow c_4 = 0 \quad \because c_2 \neq 0$ ………⑪

Using ⑪in ⑩

$\Rightarrow y(x,t) = c_2 c_3 sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$……..⑫

Using ⑤in ⑫

$\Rightarrow a\sin\frac{\pi x}{l} = c_2 c_3 sin\frac{n\pi x}{l}$

$\Rightarrow c_2 c_3 = a\,, n = 1$

using in ⑫

$\Rightarrow y(x,t) = a sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$

Note : Above example can also be solved using solution of wave equation given by equation ⑩ in section 4.4.1

$$y(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{l} + b_n sin\frac{n\pi ct}{l}\right) sin\frac{n\pi x}{l}$$

It is to be noted that boundary value conditions ②and ③have been already used in this solution.

Example 1.16 A tightly stretched string with fixed end points at $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity $\lambda x\ (l - x)$, find the displacement of the string at any distance from one end at any time t.

Solution: Let the equation of vibrating string be given by $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ …… ①

Boundary value conditions are given by :

$u(0,t) = 0$ …………②

$u(l,t) = 0$ …………③

$u(x,0) = 0$ …………④

$\frac{\partial u}{\partial t}_{t=0} = \lambda x\ (l - x)$………⑤

solution of wave equation as given by equation ⑩in section 1.4.1 is

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi ct}{l} + b_n sin\frac{n\pi ct}{l}\right) sin\frac{n\pi x}{l} \text{ .....⑥}$$

It is to be noted that boundary value conditions ②and ③ have been already used in this solution.

Now using ④ in ⑥

$$0 = \sum_{n=1}^{\infty} a_n sin\frac{n\pi x}{l}$$

$$a_n = 0 \text{ .... ⑦}$$

Using ⑦in ⑥

$$u(x,t) = \sum_{n=1}^{\infty} b_n sin\frac{n\pi ct}{l} sin\frac{n\pi x}{l} \text{ .... ⑧}$$

Now to use ⑤, differentiating ⑧ partially w.r.t t

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{l} cos\frac{n\pi ct}{l} sin\frac{n\pi x}{l}$$

Putting $t = 0$,

$$\Rightarrow \frac{\partial u}{\partial t}_{t=0} = \frac{\pi c}{l}\sum_{n=1}^{\infty} n b_n \sin\frac{n\pi x}{l} \text{ ....⑨}$$

Using ⑤in ⑨

$$\lambda x\,(l-x) = \frac{\pi c}{l}\sum_{n=1}^{\infty} n b_n \sin\frac{n\pi x}{l}$$

Multiplying both sides by $\sin\frac{n\pi x}{l}$and integrating w.r.t. x within the limits 0 to l

$$\Rightarrow \int_0^l \lambda x\,(l-x)\sin\frac{n\pi x}{l}dx = \frac{\pi c}{l} n b_n \int_0^l \sin^2\frac{n\pi x}{l}dx$$

$$= \frac{\pi c}{2l} n b_n \int_0^l (1-\cos\frac{2n\pi x}{l})dx$$

$$= \frac{\pi c}{2} n b_n$$

$$\Rightarrow \pi c n b_n = 2\int_0^l \lambda x\,(l-x)\sin\frac{n\pi x}{l}dx$$

$$= 2\lambda\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}dx$$

$$\Rightarrow \pi c n b_n = 2\lambda\Big[(lx-x^2).\Big(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\Big) - (l-2x)\Big(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\Big) + (-2)\Big(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\Big)\Big]_0^l$$

$$\Rightarrow \pi c n b_n = 2\lambda\Big[\frac{-2l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\Big]$$

$$\Rightarrow \pi c n b_n = \frac{4\lambda l^3}{n^3\pi^3}\,[1-(-1)^n]$$

$$\Rightarrow \pi c n b_n = \begin{bmatrix} \frac{8\lambda l^3}{n^3\pi^3} & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{bmatrix}$$

taking $n = 2m - 1$

$$b_n = \frac{8\lambda l^3}{C\pi^4 (2m-1)^4} \ldots\ldots\ldots⑩$$

using⑩in ⑧, required solution is

$$u(x,t) = \frac{8\lambda l^3}{C\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^4} sin\frac{(2m-1)\pi ct}{l} sin\frac{(2m-1)\pi x}{l}$$

4.4.2 Solution of heat equation using method of separation of variables

One dimensional heat flow equation is given by $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ .... ①

where $u(x,t)$is temperature function at distance $x$from origin at any time t. To solve heat equation using method of separation of variables,

Let $u = XT$ .... ②

where X is a function of x alone and T be a function of t alone.

$$\therefore \frac{\partial u}{\partial t} = XT', \frac{\partial^2 u}{\partial x^2} = X''T$$

Substituting these values of $\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}$in the heat equation given by ①

$$XT' = c^2 X''\mathrm{T}$$

Arranging in variable separable form$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T}$

Equating LHS = RHS = K $(\because X\ and\ T\ are\ independent)$

$$\Rightarrow \frac{X''}{X} = K \text{and} \frac{1}{c^2}\frac{T'}{T} = K$$

$$\Rightarrow X'' - KX = 0 \text{and} T' - Kc^2T = 0 \ldots\ldots\ldots\ldots ③$$

Solving ordinary differential equations given in ③, three cases arise

(i) When K is +ve and = p^2say

$$X'' - p^2X = 0 \text{and } T' = p^2c^2T$$

$$X = Ae^{px} + Be^{-px}, T = Ce^{p^2c^2t}$$

$$\Rightarrow u(x,t) = (Ae^{px} \ Be^{-px})Ce^{p^2c^2t}$$

$$\Rightarrow u(x,t) = ((c_1e^{px} + c_2e^{-px})e^{p^2c^2t}$$

(ii) When K is $-$ve and = $-p^2$say

$$X = Acospx + Bsinpx, \ T = Ce^{-c^2p^2t}$$

$$\Rightarrow u(x,t) = (Acospx + Bsinpx)\ Ce^{-c^2p^2t}$$

$$\Rightarrow u(x,t) = (c_1cospx + c_2sinpx)\ e^{-c^2p^2t}$$

(iii)When K = 0

$$X'' = 0 \text{and} T' = 0$$

$$\Rightarrow X = Ax + B, T = C$$

$$\Rightarrow u(x,t) = (Ax + B)C$$

$$\Rightarrow u(x,t) = c_1x + c_2$$

The solution given by (ii) i.e. corresponding to $K = -p^2$, is the most plausible solution for steady state.

Special case: When the ends of a rod are kept at $0°$C

One dimensional heat equation in steady state is given by :

$u(x,t) = (c_1cospx + c_2sinpx)\ e^{-c^2p^2t}$ ....④

Also since ends of a rod are kept at $0°$C

$u(0,t) = 0$ ....⑤

$u(l,t) = 0$ ....⑥

Using ⑤in ④⇒ 0= $C_1e^{-c^2p^2t}$

⇒ $C_1 = 0$ .... ⑦

Using ⑦ in ④ , wave equation reduces to

$u(x,t) = c_2 sinpx e^{-c^2p^2t}$........ ⑧

Now using ⑥in ⑧⇒ $0 = c_2 sinple^{-c^2p^2t}$⇒$\sin pl = 0$

$\because c_2 \neq 0$ and $e^{-c^2p^2t} \neq 0$

$\Rightarrow pl = n\pi \Rightarrow p = \frac{n\pi}{l}$, $n = 1,2,3 \ldots \ldots \ldots$⑨

using ⑨in⑧⇒ $u(x,t) = c_2 sin\frac{n\pi x}{l} e^{\frac{-c^2n^2\pi^2t}{l^2}}$

Adding up the solutions for different values of n, the most general solution is given by

$u(x,t) = \sum_{n=1}^{\infty} b_n sin\frac{n\pi x}{l} e^{\frac{-c^2n^2\pi^2t}{l^2}}$.........⑩

Example 1.17 :A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to 0°C and are kept at that temperature, find the temperature formula $u(x,t)$.

Solution: One dimensional heat flow equation is given by $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Solution of one dimensional heat equation in steady state is given by :

$u(x,t) = (c_1 cospx + c_2 sinpx)\, e^{-c^2p^2t}$ ....①

$u(0,t) = 0$

............② $u(l,t) = 0$

............③

Also initial condition is $u(x,0) = u_0$ ....④

The most general solution of heat equation① using ②and③is given by

$u(x,t) = \sum_{n=1}^{\infty} b_n sin\frac{n\pi x}{l} e^{\frac{-c^2n^2\pi^2t}{l^2}}$ .... ⑤

Using ④in ⑤⇒ $u_0 = \sum_{n=1}^{\infty} b_n sin\frac{n\pi x}{l}$ **.........**⑥

Multiplying both sides of ⑥by $sin\frac{n\pi x}{l}$, and integrating w.r.t. x within the limits 0 to l

$$\int_0^l u_0\, sin\frac{n\pi x}{l} dx = b_n \int_0^l sin^2\frac{n\pi x}{l} dx$$

$$= b_n \frac{l}{2}$$

$$\Rightarrow b_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}dx$$

$$= \frac{-2\,u_0}{l}\left[\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_0^l$$

$$b_n = \frac{2\,u_0}{n\pi}[1-(-1)^n] \quad \begin{bmatrix} \frac{4\,u_0}{n\pi}, & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{bmatrix}$$

Putting $n = 2m-1, \Rightarrow\ b_n = \frac{4\,u_0}{(2m-1)\pi}$ …………⑦

Using ⑦ in ⑤, the required temperature formula is

$$u(x,t) = \frac{4\,u_0}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\sin\frac{(2m-1)\pi x}{l}e^{\frac{-c^2(2m-1)^2\pi^2 t}{l^2}}$$

4.4.3 Solution of Laplace equation (two-dimensional heat flow) using method of separation of variables

Consider the heat flow in a uniform rectangular metal plate at any time t; if $u(x,y)$ be the temperature at time t, two dimensional heat flow equation is given by

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

In steady state, u doesn't change with and t hence $\frac{\partial u}{\partial t} = 0$

∴ Two dimensional heat flow equation in steady state is given by $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$………①

where $u(x,y)$ is temperature function at any point (x,y) of the rectangular metal plate. This is called Laplace equation in two dimensions. To solve Laplace equation using method of separation of variables,

Let $u = XY$ ………………….. ②

Where X is a function of alone x and Y be a function of along y.

$$\therefore \frac{\partial^2 u}{\partial x^2} = X''Y,\ \frac{\partial^2 u}{\partial y^2} = XY''$$

Substituting these values of $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ in the Laplace equation given by ①

$$X''Y + XY'' = 0$$

Arranging in variable separable form $\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y}$

Equating LHS = RHS = K $(\because x \text{ and } y \text{ are independent})$

$$\Rightarrow \frac{X''}{X} = K \text{ and } -\frac{Y''}{Y} = K$$

$$\Rightarrow X'' - KX = 0 \text{ and } Y'' + KY = 0 \text{ ....③}$$

Solving ordinary differential equations given in ③, three cases arise

4.4.3.1 K is +ve and = p^2 say

$$X = c_1 e^{px} + c_2 e^{-px}, Y = c_3 \cos py + c_4 \sin py$$

$$\Rightarrow u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

4.4.3.2 K is -ve and = $-p^2$ say

$$X = (c_1 \cos px + c_2 \sin px, Y = (c_3 e^{py} + c_4 e^{-py})$$

$$\Rightarrow u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

(iii)When $K = 0$

$$X'' = 0 \text{ and } Y'' = 0$$

$$\Rightarrow X = c_1 x + c_2, Y = c_3 y + c_4$$

$$\Rightarrow u(x,t) = (c_1 x + c_2)(c_3 y + c_4)$$

The solution given by (ii) i.e. corresponding to $K = -p^2$, is the most plausible solution for steady state.

Example 1.18 An infinitely long rectangular uniform plate with breadth π is bounded by two parallel edges maintained at 0°C. Base of the plate is kept at a temperature u_0 at all points. Determine the temperature at any point of the plate in the steady state.

Solution : In steady state, two dimensional heat flow equation is given by $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$....①

Boundary value conditions are

$u(0,y) = 0$............②

$u(\pi, y) = 0$............③

$\lim_{y\to\infty} u(x,y) = 0 \;,\; 0 < x < \pi$......... ④

$u(x,0) = u_0 \;,\; 0 < x < \pi$ ... ...⑤

Solution of ① is given by :

$u(x,y) = (c_1 cospx + c_2 sinpx)(\, c_3 e^{py} + \; c_4 e^{-py})$ ....⑥

Using ②in ⑥

$\Rightarrow u(0,y) = 0 = \; c_1(\, c_3 e^{py} + \; c_4 e^{-py})$

$\Rightarrow \; c_1 = 0$ .... ⑦

Using ⑦ in ⑥

$u(x,y) = \; c_2 sinpx(\, c_3 e^{py} + \; c_4 e^{-py})$..............⑧

Using ③in ⑧

$\Rightarrow u(\pi, y) = 0 = c_2 sinp\pi(\, c_3 e^{py} + \; c_4 e^{-py})$

$\Rightarrow sinp\pi = 0$

$\Rightarrow p\pi = \text{n}\pi$

$\Rightarrow p = \text{n}$ ....⑨

Using ⑨ in⑧

$\Rightarrow u(x,y) = \; c_2 sinnx(\, c_3 e^{ny} + \; c_4 e^{-ny})$ ....⑩

Using ④ in⑩

$\lim_{y\to\infty} u(x,y) = 0 = \; c_2 sinnx \lim_{y\to\infty} (\, c_3 e^{ny} + \; c_4 e^{-ny})$

$\Rightarrow \; c_3 = 0$ .... ⑪

Using ⑪ in ⑩

$\Rightarrow u(x,y) = \; c_2 c_4 sinnxe^{-ny}$

The most general solution of heat equation is given by

$u(x,t) = \sum_{n=1}^{\infty} b_n sin\, nx.\, e^{-ny}$......... ⑫where $c_2 c_4 = b_n$

Using ⑤ in ⑩

$\Rightarrow u(x,0) = u_0 = \sum_{n=1}^{\infty} b_n \sin nx$

Multiplying both sides by $\sin nx$, and integrating w.r.t. x within the limits 0 to π

$$b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx \, dx$$

$$\Rightarrow b_n = \frac{2u_0}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \begin{bmatrix} \frac{4\,u_0}{n\pi}, & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{bmatrix} \ldots\ldots ⑬$$

Let $n = 2m - 1$ as n is odd

Using ⑬ in ⑫ the required temperature formula is:

$$u(x,t) = \frac{4\,u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2m-1)}\sin(2m-1)x.e^{-(2m-1)y}$$

Chapter 5

LAPLACE TRANSFORM

5.1 Introduction

A transformation is an operation which converts a mathematical expression to a different but equivalent form. The well-known transformation logarithms reduce multiplication and division to a simpler process of addition subtraction.

The Laplace transform is a powerful mathematical technique which solves linear equations with given initial conditions by using algebra methods. The Laplace transform can also be used to solve systems of differential equations, Partial differential equations and integral equations. In this chapter, we will discuss about the definition, properties of Laplace transform and derive the transforms of some functions which usually occur in the solution of linear differential equations.

5.2 Laplace transform

Let $f(t)$ be a function of t defined for all $t \geq 0$ .then the Laplace transform of$f(t)$, denoted by $L[\,f(t)]$ is defined by

$$L[\,f(t)] = \int_0^\infty e^{-st}\, f(t)dt$$

Provided that the integral exists, "s" is a parameter which may be real or complex. Clearly $L[f(t)]$ is a function of s and is briefly written as $F(s)$ $(i.e.)$ $L[\,f(t)] = F(s)$

Piecewise continuous function

A function $f(t)$ is said to be piecewise continuous is an interval $a \leq t \leq b$, if the interval can be sub divided into a finite number of intervals in each of which the function is continuous and has finite right and left-hand limits.

Exponential order

A function $f(t)$ is said to be exponential order if $\lim_{t\to\infty} e^{-st}f(t)$ is a finite quantity, where $s > 0$(exists).

Example: 5.1 Show that the function $f(t) = e^{t^3}$ is not of exponential order.

Solution:

$$\lim_{t\to\infty} e^{-st}\, e^{t^3} = \lim_{t\to\infty} e^{-st+t^3} = \lim_{t\to\infty} e^{t^3-st}$$

$$= e^{\infty} = \infty, \text{ not a finite quantity.}$$

Hence $f(t) = e^{t^3}$ is not of exponential order.

Sufficient conditions for the existence of the Laplace transform

The Laplace transform of $f(t)$ exists if

i) $f(t)$ is piecewise continuous in the interval $a \leq t \leq b$

ii) $f(t)$ is of exponential order.

Note: The above conditions are only sufficient conditions and not a necessary condition.

Example: 5.2 Prove that Laplace transform of e^{t^2} does not exist.

Solution:

$$\lim_{t\to\infty} e^{-st}\, e^{t^2} = \lim_{t\to\infty} e^{-st+t^2} = \lim_{t\to\infty} e^{t^2-st}$$

$= e^{\infty} = \infty$,not a finite quantity.

$\therefore e^{t^2}$ is not of exponential order.

Hence Laplace transform of e^{t^2} does not exist.

5.3 Properties of Laplace transform

Property: 1 Linear property

$\boldsymbol{L[af(t) \pm bg(t)] = aL[f(t)] \pm bL[g(t)]}$ **, where a and b are constants.**

Proof:

$$L[af(t) \pm bg(t)] = \int_0^{\infty} [af(t) \pm bg(t)]\, e^{-st} dt$$

$$= a\int_0^{\infty} f(t) e^{-st} dt \pm b\int_0^{\infty} g(t)\, e^{-st} dt$$

$$\boxed{L[af(t) \pm bg(t)] = a\, L[f(t)] \pm b\, L[g(t)]}$$

Property: 2 Change of scale property.

If $\boldsymbol{L[f(t)] = F(s)}$**, then** $\boldsymbol{L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)}$; $\boldsymbol{a} > 0$

Proof:

Given $L[f(t)] = F(s)$

$$\therefore \int_0^{\infty} e^{-st} f(t)\, dt = F(s) \cdots\cdots (1)$$

By the definition of Laplace transform, we have

$$L[f(at)] = \int_0^{\infty} e^{-st} f(at)\, dt \cdots\cdots (2)$$

Put at= x $ie., t = \frac{x}{a} \Rightarrow dt = \frac{dx}{a}$

$$(2) \Rightarrow L[f(at)] = \int_0^{\infty} e^{\frac{-sx}{a}} f(x) \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^{\infty} e^{\frac{-sx}{a}} f(x) dx$$

Replace x by t, $L[f(at)] = \frac{1}{a} \int_0^{\infty} e^{\frac{-st}{a}} f(t) dt$

$$\boxed{\boldsymbol{L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)} \ ; \boldsymbol{a} > 0}$$

Property: 3 First shifting property.

If $\boldsymbol{L[f(t)] = F(s)}$**, then i)** $\boldsymbol{L[e^{-at} f(t)] = F(s + a)}$

ii) $\boldsymbol{L[e^{at} f(t)] = F(s - a)}$

Proof:

(i) $L[e^{-at} f(t)] = F(s + a)$

Given $L[f(t)] = F(s)$

$$\therefore \int_0^{\infty} e^{-st} f(t)\, dt = F(s) \cdots (1)$$

By the definition of Laplace transform, we have

$$L[e^{-at}f(at)] = \int_0^{\infty} e^{-st}\, e^{-at}f(t)\, dt$$

$$= \int_0^{\infty} e^{-(s+a)t} f(t)\, dt$$

$$= F(s+a) \quad \text{by (1)}$$

(ii) $L[e^{at}f(at)] = \int_0^{\infty} e^{-st}\, e^{at}f(t)\, dt$

$$= \int_0^{\infty} e^{-(s-a)t} f(t)\, dt$$

$$= F(s-a) \quad \text{by (1)}$$

Property: 4 Laplace transforms of derivatives $L[f'(t)] = sL[f(t)] - f(0)$

Proof:

$$L[f'(t)] = \int_0^{\infty} e^{-st} f'(t)\, dt = \int_0^{\infty} u dv$$

$$= [uv]_0^{\infty} - \int_0^{\infty} v du$$

$$= [e^{-st} f(t)]_0^{\infty} - \int_0^{\infty} f(t)\, (-s)e^{-st}\, dt$$

$$= 0 - f(0) + sL[f(t)]$$

$$= sL[f(t)] - f(0)$$

$$\boldsymbol{L[f'(t)] = sL[f(t)] - f(0)}$$

$u = e^{-st}$
$\therefore du = -se^{-st}dt$
$dv = f'(t)dt$
$\therefore v = \int f'(t)dt$
$= f(t)$

Property: 5 Laplace transform of derivative of order n

$$\boldsymbol{L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2}f'(0) \cdots - s^{n-3} f''(0) - \cdots f^{n-1}(0)}$$

Proof:

We know that $L[f'(t)] = sL[f(t)] - f(0) \cdots\cdots (1)$

$$L[f^n(t)] = L[[f'(t)]']$$

$$= sL[f'(t)] - f'(0)$$

$$= s[sL[f(t)] - f(0)] - f'(0)$$

$$= s^2L[f(t)] - sf(0) - f'(0)$$

Similarly, $L[f'''(t)] = s^3L[f(t)] - s^2f(0) - sf'(0) - f''(0)$

In general, $\boldsymbol{L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2}f'(0) \cdots - s^{n-3} f''(0) - \cdots f^{n-1}(0)}$

Laplace transform of integrals

Theorem: 1 If $L[f(t)] = F(s)$, then $L \int_0^t f(t)\, dt] = \frac{F(s)}{s}$

Proof:

Let, $g(t) = \int_0^t f(t)dt$

$\therefore g'(t) = f(t)$

And $g(0) = \int_0^0 f(t)dt = 0$

$$\text{Now} L[g'(t)] = L[f(t)]$$

$$sL[g(t)] - g(0) = L[f(t)]$$

$$sL[g(t)] = L[f(t)] \quad \therefore g(0) = 0$$

$$L[g(t)] = \frac{L[f(t)]}{s}$$

$$\therefore \quad \int_0^t f(t)\,dt = \frac{F(s)}{s}$$

Theorem: 2 If $L[f(t)] = F(s)$, then $L[tf(t)] = -\dfrac{d}{ds}F(s)$

Proof:

Given $L[f(t)] = F(s)$

$$\therefore \int_0^\infty e^{-st} f(t)\,dt = F(s) \cdots\cdots (1)$$

Differentiating (1) with respect to s, we get

$$\frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \text{F(s)}$$

$$\int_0^\infty \frac{\partial}{\partial s}(e^{-st})f(t)\,dt = \frac{d}{ds}F(s)$$

$$\int_0^\infty (-t)e^{-st} f(t)\,dt = \frac{d}{ds}\ F(s)$$

$$-\int_0^\infty e^{-st} f(t)\,dt = \frac{d}{ds}F(s)$$

$$-L[tf(t)] = \frac{d}{ds}F(s)$$

$$\therefore L[tf(t)] = -\frac{d}{ds}F(s)$$

Note: In general $L[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n}F(s)$

Example: 5.3 If $L[f(t)] = \dfrac{s^2-s+1}{(2s+1)^2(s-1)}$ then find $L[f(2t)]$.

Solution:

$$\text{Given } L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)} = F(s)$$

$$L[f(2t)] = \frac{1}{2}F\left(\frac{s}{2}\right)$$

$$= \frac{1}{2}\frac{\left(\frac{s}{2}\right)^2 - \frac{s}{2}+1}{\left(2\frac{s}{2}+1\right)^2\left(\frac{s}{2}-1\right)}$$

$$= \frac{1}{2}\frac{\left[\frac{s^2}{4} - \frac{s}{2}+4\right]}{(s+1)^2\left(\frac{s-2}{2}\right)}$$

$$= \frac{s^2-2s+1}{4(s+1)^2(s-2)}$$

Laplace transform of some Standard functions

Result: 1 Prove that $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[t^n] = \int_0^\infty e^{-st}\, t^n\, dt$$

$$L[t^n] = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \int_0^\infty e^{-u} \frac{u^n}{s^{n+1}}\, du$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n\, du$$

$$\therefore L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \qquad \because \int_0^\infty e^{-u} u^n\, du$$

Let $st = u \cdots\cdots (1)$

$t = \frac{u}{s}$

$dt = \frac{du}{s}$

When $t \to 0 (1) \Rightarrow u \to 0$

,

$t \to \infty, (1) \Rightarrow u \to \infty$

Result: 2 Prove that $L(e^{at}) = \frac{1}{s-a}, s > a$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$\therefore L(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-t(s-a)} f(t)\, dt$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty$$

$$= -\left[0 - \left(\frac{1}{s-a}\right)\right]$$

$$\therefore L(e^{at}) = \frac{1}{s-a}$$

Result: 3 Prove that $L(e^{-at}) = \frac{1}{s+a}, s > a$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$\therefore L(e^{-at}) = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-t(s+a)} f(t)\, dt$$

$$= \left[\frac{e^{-t(s+a)}}{-(s+a)}\right]_0^\infty$$

$$= -\left[0 - \left(\frac{1}{s+a}\right)\right]$$

$$\therefore L(e^{at}) = \frac{1}{s+a}$$

Result: 4 Prove that $L[sinat] = \frac{a}{s^2+a^2}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[sinat] = \int_0^\infty e^{-st} sinat\, dt$$

$$\therefore L[cosat] = \frac{s}{s^2+a^2}, s > |a| \qquad \because \int_0^\infty e^{-at} cosbt\, dt = \frac{a}{a^2+b^2}$$

Result: 5 Prove that $L[cosat] = \frac{s}{s^2+a^2}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[cosat] = \int_0^\infty e^{-st} cosat\, dt$$

$$\therefore L[cosat] = \frac{s}{s^2+a^2}, \quad s > |a|, \qquad \because \int_0^\infty e^{-at} cosbt\, dt = \frac{a}{a^2+b^2}$$

Result: 6 Prove that $L[sinhat] = \frac{a}{s^2-a^2}$ $, s > |a|$

Proof:

We have $L[sinhat] = L\left[\frac{e^{at}-e^{-at}}{2}\right]$

$$= \frac{1}{2}[L(e^{at}) - L(e^{-at})]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2} \frac{s+a-s+a}{s^2-a^2}$$

$$= \frac{1}{2} \frac{2a}{s^2-a^2}$$

$$\therefore L[sinhat] = \frac{a}{s^2-a^2} \quad , s > |a|$$

Result: 7 Prove that $L[coshat] = \frac{s}{s^2-a^2}$ $, s > |a|$

Proof:

We have $L[coshat] = L\left[\frac{e^{at}+e^{-at}}{2}\right]$

$$= \frac{1}{2}[L(e^{at}) + L(e^{-at})]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2} \frac{s+a+s-a}{s^2-a^2}$$

$$= \frac{1}{2} \frac{2s}{s^2-a^2}$$

$$\therefore L[coshat] = \frac{s}{s^2-a^2} \quad , s > |a|$$

Example: 5.4 Find L[$t^{1/2}$].

Solution:

We have $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Put $n = \frac{1}{2}$

$$\therefore L\left[t^{\frac{1}{2}}\right] = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} \qquad \because \Gamma(n+1) = n\Gamma n$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}+1}} \qquad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$$

$$\therefore L\left[t^{\frac{1}{2}}\right] = \frac{\sqrt{\pi}}{2s\sqrt{s}}$$

Example: 5.5 Find the Laplace transform of $t^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{t}}$

Solution:

We have $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Put $n = -\frac{1}{2}$

$$\therefore L\left[t^{-\frac{1}{2}}\right] = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}} \qquad \because \Gamma(n+1) = n\Gamma n$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} \qquad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$\therefore L\left[\frac{1}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}$$

Result: 8 Prove that $L[1] = \frac{1}{s}$.

Solution:

$$L[1] = \int_0^\infty e^{-st} \cdot 1 \cdot dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \left(0 - \frac{1}{-s}\right) = \frac{1}{s}$$

$$L[1] = \frac{1}{s}.$$

FORMULA

$L[f(t)] = F(s)$	$L[f(t)] = F(s)$
$L[1] = \frac{1}{s}$	$L[sinat] = \frac{a}{s^2 + a^2}$
$L[t] = \frac{1}{s^2}$	$L[cosat] = \frac{s}{s^2 + a^2}$
$L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ **if n is not an integer**	$L[coshat] = \frac{s}{s^2 - a^2}$
$L[t^n] = \frac{n!}{s^{n+1}}$ **if n is an integer**	$L[sinhat] = \frac{a}{s^2 - a^2}$
$L(e^{at}) = \frac{1}{s-a}$	
$L(e^{at}) = \frac{1}{s+a}$	

Problems using Linear property

Example: 6 Find the Laplace transform for the following

i. $3t^2 + 2t + 1$	vi. $sin(at + b)$
ii. $(t + 2)^3$	vii. $cos^3 2t$
iii. a^t	viii. $sin^3 t$
iv. $e^{2t} + 3$	ix. $sin^2 t$
v. $sin\sqrt{2}\, t$	x. $cos^2 2t$
	xi. $cos5tcos4t$

Solution:

(i) Given $f(t) = 3t^2 + 2t + 1$

$$L[f(t)] = L[3t^2 + 2t + 1]$$

$$= L[3t^2] + L[2t] + L[1]$$

$$= L[3t^2] + L[2t] + L[1]$$

$$= 3L[t^2] + 2L[t] + L[1]$$

$$= 3\frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s}$$

$$\therefore L[3t^2 + 2t + 1] = \frac{6}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

(ii) Given $f(t) = (t + 2)^3 = t^3 + 3t^2(2) + 3t2^2 + 2^3$

$$L[f(t)] = L[t^3 + 3t^2(2) + 3t2^2 + 2^3]$$

$$= L[t^3] + L[6t^2] + L[12t] + L[8]$$

$$= \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{12}{s}$$

(iii) Given $f(t) = a^t$

$$L[f(t)] = L[a^t] = L[e^{tloga}]$$

$$L[a^t] = \frac{1}{s-loga}$$

(iv) Given $f(t) = e^{2t+3}$

$$L[f(t)] = L[e^{2t+3}] = L[e^{2t}.e^3]$$

$$= e^3 L[e^{2t}]$$

$$= e^3 [\frac{-1}{s-2}]$$

$$\therefore L[e^{2t+3}] = e^3 [\frac{1}{s-2}]$$

(v) $L[sin\sqrt{2}t] = \frac{\sqrt{2}}{s^2+2}$

(vi) Given $f(t) = \sin(at + b) = sinatcosb + cosatsinb$

$$L[f(t)] = L[\sin(at + b)]$$

$$= L[sinatcosb + cosatsinb]$$

$$= cosb\, L[\sin at] + sinb\, L[\cos at]$$

$$L[\sin(at + b)] = cosb\, \frac{s}{s^2+a^2} + sinb\, \frac{s}{s^2+a^2}$$

(vii) Given $f(t) = \cos^3 2t = \frac{1}{4}[3cos2t + cos6t]$

$$\because \cos^3\theta = \frac{3cos\theta + cos3\theta}{4}$$

$$L[f(t)] = \frac{1}{4} L[3cos2t + cos6t]$$

$$= \frac{1}{4}[3L(cos2t) + L(cos6t)]$$

$$= \frac{1}{4}[3\frac{s}{s^2+4} + \frac{s}{s^2+36}]$$

$$L[\cos^3 2t] = \frac{1}{4}[3\frac{s}{s^2+4} + \frac{s}{s^2+36}]$$

(viii) Given $f(t) = \sin^3 t = \frac{1}{4}[3sint - sin3t]$

$$L[f(t)] = \frac{1}{4}L[3sint - sin3t]$$

$$= \frac{1}{4}[3L(sint) - L(sin3t)]$$

$$= \frac{1}{4}[3\frac{1}{s^2+1} - \frac{3}{s^2+9}]$$

$$L[\sin^3 t] = \frac{3}{4}[\frac{1}{s^2+1} - \frac{1}{s^2+9}]$$

(ix) Given $f(t) = \sin^2 t = \frac{1-cos2t}{2}$

$$L[f(t)] = L[\frac{1-cos2t}{2}]$$

$$= \frac{1}{2}[L(1) - L(cos2t)]$$

$$= \frac{1}{2}[\frac{1}{s} - \frac{s}{s^2+4}]$$

$$L[\cos^2 2t] = \frac{1}{2}[\frac{1}{s} - \frac{s}{s^2+4}]$$

(x) Given $f(t) = \cos^2 2t = \frac{1+cos4t}{2}$

$$L[f(t)] = L[\frac{1+cos4t}{2}]$$

$$= \frac{1}{2}[L(1) + L(cos4t)]$$

$$= \frac{1}{2}[\frac{1}{s} + \frac{s}{s^2+16}]$$

$$L[\cos^2 2t] = \frac{1}{2}[\frac{1}{s} + \frac{s}{s^2+16}]$$

(xi) Given $f(t) = cos5tcos4t$

$$L[f(t)] = L[cos5tcos4t]$$

$$= \frac{1}{2}[L(cos9t) + L(cost)]$$

$$= \frac{1}{2}[\frac{s}{s^2+81} + \frac{s}{s^2+1}]$$

Problems using First Shifting theorem

$$\boldsymbol{L[e^{-at}f(t)] = L[f(t)]_{s\to s+a}}$$

$$\boldsymbol{L[e^{at}f(t)] = L[f(t)]_{s\to s-a}}$$

Example: 5.7 Find the Laplace transform for the following:

i. te^{-3t} ii. t^3e^{2t} iii. $e^{4t}sin2t$ iv. $e^{-5t}cos3t$	v. $sinh2tcos3t$ vi. $t^2 2^t$ vii. $t^3 2^{-t}$ viii. $e^{-2t}sin3tcos2t$	ix. $e^{-3t}cos4tcos2t$ x. $e^{4t}cos3tsin2t$ xi. cosh3tsin2t

(i) $\boldsymbol{te^{-3t}}$

$$L[te^{-3t}] = L[t]_{s\to s+3}$$

$$= \left(\frac{1}{s^2}\right)_{s\to s+3} \qquad \because L(t) = \frac{1}{s^2}$$

$$\therefore L[te^{-3t}] = \frac{1}{(s+3)^2}$$

(ii) $\boldsymbol{t^3e^{2t}}$

$$L[t^3e^{2t}] = L[t^3]_{s\to s-2}$$

$$= \left(\frac{3!}{s^4}\right)_{s\to s-2} \qquad \because L(t) = \frac{3!}{s^{3+1}}$$

$$\therefore L[t^3e^{2t}] = \frac{6}{(s-2)^4}$$

(iii) $\boldsymbol{e^{4t}sin2t}$

$$L[e^{4t}sin2t] = L[sin2t]_{s\to s-4}$$

$$= \left(\frac{2}{s^2+2^2}\right)_{s\to s-4}$$

$$= \frac{2}{(s-4)^2+4}$$

$$= \frac{2}{s^2-8s+16+4}$$

$$\therefore L[e^{4t}sin2t] = \frac{2}{s^2-8s+20}$$

(iv) $\boldsymbol{L[e^{-5t}cos3t]}$

$$L[e^{-5t}cos3t] = L[cos3t]_{s\to s+5}$$

$$= \left(\frac{s}{s^2+3^2}\right)_{s\to s+5}$$

$$= \frac{s+5}{(s+5)^2+9}$$

$$= \frac{s+5}{s^2+10s+25+9}$$

$$\therefore L[e^{-5t}cos3t] = \frac{s+5}{s^2+10s+34}$$

(v) $\boldsymbol{L[sinh2tcos3t]}$

$$L[sinh2tcos3t] = L\left[\left(\frac{e^{2t}-e^{-2t}}{2}\right) cos3t\right]$$

$$= \frac{1}{2}\left[L(e^{2t}cos3t) - L(e^{-2t}cos3t)\right]$$

$$= \frac{1}{2}\left[L(cos3t)_{s\to s-2} - L(cos3t)_{s\to s+2}\right]$$

$$= \frac{1}{2}\left[\left(\frac{s}{s^2+3^2}\right)_{s\to s-2} - \left(\frac{s}{s^2+3^2}\right)_{s\to s+2}\right]$$

$$\therefore L[sinh2tcos3t] = \frac{1}{2}\left[\frac{s-2}{(s-2)^2+9} - \frac{s+2}{(s+2)^2+9}\right]$$

(vi) $\boldsymbol{L[cosh3tsin2t]}$

$$L[cosh3tsin2t] = L\left[\left(\frac{e^{3t}+e^{-3t}}{2}\right) sin2t\right]$$

$$= \frac{1}{2}[L(e^{3t}sin2t) + L(e^{-3t}sin2t)]$$

$$= \frac{1}{2}[L(sin2t)_{s\to s-3} + L(sin2t)_{s\to s+3}]$$

$$= \frac{1}{2}[(\frac{2}{s^2+2^2})_{s\to s-3} + (\frac{2}{s^2+2^2})_{s\to s+3}]$$

$$\therefore\ L[cosh3tsin2t] = \frac{1}{2}[\frac{2}{(s-3)^2+4} + \frac{2}{(s+3)^2+4}]$$

(vii) $t^2 2^t$

$$L[t^2 2^t] = L[t^2 e^{log2^t}]$$

$$= L[t^2 e^{tlog2}] = L[t^2]_{s\to s-log2}$$

$$= (\frac{2!}{s^3})_{s\to s-log2}$$

$$= \frac{2}{(s-log2)^3}$$

$$\therefore L[t^2 2^t] = \frac{2}{(s-log2)^3}$$

(viii) $t^3 2^{-t}$

$$L[t^3 2^{-t}] = L[t^3 e^{log2^{-t}}]$$

$$= L[t^3 e^{-tlog2}] = L[t^3]_{s\to s+log2}$$

$$= (\frac{3!}{s^4})_{s\to s+log2}$$

$$= \frac{6}{(s+log2)^4}$$

$$\therefore L[t^3 2^{-t}] = \frac{6}{(s+log2)^4}$$

(ix) $L[e^{-2t}sin3tcos2t]$

$$L[e^{-2t}sin3tcos2t] = L[sin3tcos2t]_{s\to s+2}$$

$$= \frac{1}{2}L[\sin(3t+2t) + \sin(3t-2t)]_{s\to s+2}$$

$$= \frac{1}{2}L[\sin 5t + \sin t]_{s\to s+2}$$

$$= \frac{1}{2}[\mathrm{L}(\sin 5t) + L(\sin t)]_{s\to s+2}$$

$$= \frac{1}{2}[\frac{5}{s^2+5^2} + \frac{1}{s^2+1^2}]_{s\to s+2}$$

$$= \frac{1}{2}[\frac{5}{(s+2)^2+25} + \frac{1}{(s+2)^2+1}]$$

$$\therefore\ L[e^{-2t}sin3tcos2t] == \frac{1}{2}[\frac{5}{(s+2)^2+25} + \frac{1}{(s+2)^2+1}]$$

(x) $L[e^{-3t}cos4tcos2t]$

$$L[e^{-3t}cos4tcos2t] = L[cos4tcos2t]_{s\to s+3}$$

$$= \frac{1}{2}L[\cos(4t+2t) + \cos(4t-2t)]_{s\to s+3}$$

$$= \frac{1}{2}L[\cos 6t + \cos 2t]_{s\to s+3}$$

$$= \frac{1}{2}[\mathrm{L}(\cos 6\,t) + L(\cos 2\,t)]_{s\to s+3}$$

$$= \frac{1}{2}[\frac{s}{s^2+6^2} + \frac{s}{s^2+2^2}]_{s\to s+3}$$

$$= \frac{1}{2}[\frac{s+3}{(s+3)^2+36} + \frac{s+3}{(s+3)^2+4}]$$

$$\therefore L[e^{-3t}cos4tcos2t] = \frac{1}{2}[\frac{s+3}{(s+3)^2+36} + \frac{s+3}{(s+3)^2+4}]$$

(xi) $L[e^{4t}cos3tsin2t]$

$$L[e^{4t}cos3tsin2t] = L[cos3tsin2t]_{s\to s-4}$$

$$= \frac{1}{2}L[\sin(3t+2t) - \sin(3t-2t)]_{s\to s-4}$$

$$= \frac{1}{2}L[\sin 5t - \sin t]_{s\to s-4}$$

$$= \frac{1}{2}[\mathrm{L}(\sin 5t) - L(\sin t)]_{s\to s-4}$$

$$= \frac{1}{2}[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}]_{s\to s-4}$$

$$= \frac{1}{2}[\frac{5}{(s-4)^2+25} + \frac{1}{(s-4)^2+1}]$$

$$\therefore L[e^{4t}cos3tsin2t] = \frac{1}{2}[\frac{5}{(s-4)^2+25} + \frac{1}{(s-4)^2+1}]$$

Exercise: 5.1

Find the Laplace transform for the following

1. $\cos^2 3t$ — **Ans:** $\frac{1}{4}[\frac{3s}{s^2+9} + \frac{s}{s^2+81}]$

2. $sin3tcos4t$ — **Ans:** $\frac{1}{4}[\frac{7}{s^2+49} - \frac{1}{s^2+1}]$

3. te^{2t} — **Ans:** $\frac{1}{(s-2)^2}$

4. t^4e^{-3t} — **Ans:** $\frac{4!}{(s-3)^5}$

5. $e^{4t}sin2t$ — **Ans:** $\frac{2}{(s-4)^2+4}$

6. $e^{-5t}cos3t$ — **Ans:** $\frac{s+5}{(s+5)^2+9}$

7. $t^3 3^t$ — **Ans:** $\frac{3!}{(s-log3)^4}$

8. $t^5 4^{-t}$ — **Ans:** $\frac{5!}{(s+log4)^6}$

9. $e^{-2tsin3tcos2t}$ — **Ans:** $\frac{5}{(s+2)^2+25} + \frac{1}{(s+2)^2+1}$

10. $e^{-3tcos4tcos2t}$ — **Ans:** $\frac{s+3}{(s+3)^2+36} + \frac{s+3}{(s+3)^2+4}$

11. $sinhtsin4t$ — **Ans:** $\frac{4}{(s-1)^2+16} - \frac{4}{(s+1)^2+16}$

12. $cosh2tcos2t$ — **Ans:** $\frac{1}{2}[\frac{s-2}{(s-2)^2+4} - \frac{s+2}{(s+2)^2+4}]$

5.4 Laplace transform of derivatives and integrals

Problems using the formula

$$\boldsymbol{L[tf(t)] = \frac{-d}{ds} L[f(t)]}$$

Example: 5.8 Find the Laplace transform for $\boldsymbol{tsin4t}$

Solution:

$$L[tsin4t] = \frac{-d}{ds} L[tsin4t]$$

$$= \frac{-d}{ds}\left[\frac{4}{s^2+4}\right]$$

$$= \frac{-[(s^2+16)0-4(2s)]}{(s^2+16)^2}$$

$$\therefore L[tsin4t] = \frac{8s}{(s^2+16)^2}$$

Example: 5.9 Find $\boldsymbol{L[tsin^2t]}$

Solution:

$$L[tsin^2t] = \frac{-d}{ds} L[\sin^2 t] = \frac{-d}{ds} L\left[\frac{(1-cos2t)}{2}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}[L(1) - L(cos2t)]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} - \frac{\boldsymbol{s}}{\boldsymbol{s^2+4}}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{\boldsymbol{s^2+4-s^2}}{\boldsymbol{s(s^2+4)}}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{\boldsymbol{4}}{\boldsymbol{s(s^2+4)}}\right]$$

$$= -\frac{4}{2}\frac{d}{ds}\left[\frac{\boldsymbol{1}}{\boldsymbol{s(s^2+4)}}\right]$$

$$= -2\left[\frac{\boldsymbol{0-(3s^2+4)}}{\boldsymbol{(s^3+4s)^2}}\right]$$

$$\therefore L[tsin^2t] = \frac{\boldsymbol{2(3s^2+4)}}{\boldsymbol{(s^3+4s)^2}}$$

Example: 5.10 Find $\boldsymbol{L[tcos^2 2t]}$

Solution:

$$L[cos^2 2t] = \frac{-d}{ds} L[\cos^2 2t] = \frac{-d}{ds} L\left[\frac{(1+cos4t)}{2}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}[L(1) + L(cos4t)]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} + \frac{\boldsymbol{s}}{\boldsymbol{s^2+16}}\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} + \frac{\boldsymbol{(s^2+16)1-s.2s}}{\boldsymbol{(s^2+16)^2}}\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} + \frac{\boldsymbol{s^2+16-2s^2}}{\boldsymbol{(s^2+16)^2}}\right]$$

$$\therefore L[cos^2 2t] = \frac{1}{2}\left[\frac{1}{s^2} - \frac{\boldsymbol{16-s^2}}{\boldsymbol{(s^2+16)^2}}\right]$$

Example: 5.11 Find the Laplace transform for $tsinh2t$

Solution:

$$L[sinh2t] = \frac{-d}{ds}L[sinh2t]$$

$$= \frac{-d}{ds}\left[\frac{2}{s^2-4}\right]$$

$$= \frac{-[(s^2-4)\,0 - 2(2s)]}{(s^2-4)^2}$$

$$\therefore L[tsinh2t] = \frac{4s}{(s^2-4)^2}$$

Example: 5.12 Find the Laplace transform for $f(t) = sinat - atcosat$

Solution:

$$L[sinat - atcosat] = \mathrm{L}(\sin at) - a\,\mathrm{L(tcosat)}$$

$$= \frac{a}{s^2+a^2} - a\left(\frac{-d}{ds}L[cosat]\right)$$

$$= \frac{a}{s^2+a^2} + a\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{(s^2+a^2)1 - s(2s)}{(s^2+a^2)^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{s^2+a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{a(s^2+a^2)+a(a^2-s^2)}{(s^2+a^2)^2}$$

$$= \frac{as^2+a^3+a^3-as^2)}{(s^2+a^2)^2}$$

$$\therefore L[sinat - atcosat] = \frac{2a^3}{(s^2+a^2)^2}$$

Example: 5.13 Find the Laplace transform for the following

(i) $te^{-3t}sin2t$ (ii) $te^{-t}cosat$ 9iii) $tsinhtcos2t$

Solution:

(i) $L[te^{-3t}sin2t] = L[tsin2t]_{s\to s+3} = \frac{-d}{ds}L[sin2t]_{s\to s+3}$

$$= \frac{-d}{ds}\left(\frac{2}{s^2+2^2}\right)_{s\to s+3}$$

$$= \left[\frac{(s^2+4)0-2(2s)}{(s^2+4)^2}\right]_{s\to s+3}$$

$$= \left[\frac{4s}{(s^2+4)^2}\right]_{s\to s+3}$$

$$\therefore \quad L[te^{-3t}sin2t] = \frac{4(s+3)}{((s+3)^2+4\,)^2}$$

(ii) $L[te^{-t}cosat] = L[tcosat]_{s\to s+1} = \frac{-d}{ds}L[cosat]_{s\to s+1}$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+a^2}\right)_{s\to s+1}$$

$$= -\left[\frac{(s^2+a^2)1-s(2s)}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$= -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$= \left[\frac{s^2-a^2}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$\therefore \quad L[te^{-t}cosat] = \frac{(s+1)^2-a^2}{((s+1)^2+a^2\ )^2}$$

(iii) $L[tsinhtcos2t]$

$$L[tsinhtcos2t] = L\left[t\left(\frac{e^t-e^{-t}}{2}\right)cos2t\right]$$

$$= \frac{1}{2}[L(te^tcos2t) - L(te^{-t}cos2t)]$$

$$= \frac{1}{2}\left[\frac{-d}{ds}L[cos2t]_{s\to s-1} + \frac{d}{ds}L[cos2t]_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[\frac{-d}{ds}\left(\frac{s}{s^2+4}\right)_{s\to s-1} + \frac{d}{ds}\left(\frac{s}{s^2+4}\right)_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[-\left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s\to s-1} + \left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[-\left[\frac{4-s^2}{(s^2+4)^2}\right]_{s\to s-1} + \left[\frac{4-s^2}{(s^2+4)^2}\right]_{s\to s+1}\right]$$

$$\therefore L[tsinhtcos2t] = \frac{1}{2}\left[\frac{(s-1)^2-4}{((s-1)^2+4\ )^2} + \frac{4-(s+1)^2}{((s+1)^2+4\ )^2}\right]$$

Problems using the formula

$$\boldsymbol{L[t^2f(t)] = \frac{d^2}{ds^2}L[f(t)]}$$

Example: 5.14 Find the Laplace transform for (i) $\boldsymbol{t^2sint}$ **(ii)** $\boldsymbol{t^2cos2t}$

Solution:

(i) $L[t^2sint] = \frac{d^2}{ds^2}L[sint]$

$$= \frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{d}{ds}\ \frac{[(s^2+1)0-1(2s)]}{(s^2+1)^2}$$

$$= \frac{d}{ds}\ \frac{-2s}{(s^2+1)^2}$$

$$= -2\frac{d}{ds}\left(\frac{s}{(s^2+1)^2}\right)$$

$$= \frac{-2[(s^2+1)^2(1)-s(2)(s^2+1)(2s)]}{(s^2+1)^4}$$

$$= \frac{-2(s^2+1)[(s^2+1)-4s^2]}{(s^2+1)^4}$$

$$= \frac{-2[1-3s^2]}{(s^2+1)^3}$$

$$\therefore L[t^2sint] = \frac{6s^2-2}{(s^2+1)^3}$$

(ii) $L[t^2cos2t] = \frac{d^2}{ds^2}L[cos2t]$

$= \frac{d^2}{ds^2}[\frac{s}{s^2+4}]$

$= \frac{d}{ds}(\frac{[(s^2+4)1-s(2s)]}{(s^2+4)^2}$

$= \frac{d}{ds}(\frac{4-s^2}{(s^2+4)^2})$

$= \frac{[(s^2+4)^2(-2s) -(4-s^2)2(s^2+4)(2s)]}{(s^2+4)^4}$

$= \frac{2s(s^2+4)[(s^2+4)(-1)-(4-s^2)2]}{(s^2+4)^4}$

$= \frac{2s[s^2-12]}{(s^2+4)^3}$

$\therefore L[t^2\cos 2t] = \frac{2s[s^2-12]}{(s^2+4)^3}$

Example: 5.15 Find the Laplace transform for (i) $t^2e^{-2t}cost$ (ii) $t^2e^{4t}sin3t$

Solution:

(i) $L[t^2e^{-2t}cost] = L[t^2cost]_{s\to s+2} = \frac{d^2}{ds^2}L[cost]_{s\to s+2}$

$= \frac{d^2}{ds^2}(\frac{s}{s^2+1})_{s\to s+2}$

$= \frac{d}{ds}[\frac{(s^2+1)1-s(2s)}{(s^2+1)^2}]_{s\to s+2}$

$= \frac{d}{ds}[\frac{1-s^2}{(s^2+1)^2}]_{s\to s+2}$

$= [\frac{[(s^2+1)^2(-2s) -(1-s^2)2(s^2+1)(2s)]}{(s^2+1)^4}]_{s\to s+2}$

$= (s^2+1)[\frac{[(s^2+1)(-2s)-4s(1-s^2)]}{(s^2+1)^4}]_{s\to s+2}$

$= [\frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}]_{s\to s+2}$

$= [\frac{2s^3-6s}{(s^2+1)^3}]_{s\to s+2}$

$\therefore L[t^2e^{-2t}cost] = \frac{2(s+2)^3-6(s+2)}{((s+2)^2+1)^3}$

(ii) $L[t^2e^{4t}sin3t] = L[t^2sin3t]_{s\to s-4} = \frac{d^2}{ds^2}L[sin3t]_{s\to s-4}$

$= \frac{d^2}{ds^2}(\frac{3}{s^2+9})_{s\to s-4}$

$= \frac{d}{ds}[\frac{(s^2+9)0-3(2s)}{(s^2+9)^2}]_{s\to s-4}$

$= \frac{d}{ds}[\frac{-6s}{(s^2+9)^2}]_{s\to s-4} = -6\frac{d}{ds}[\frac{s}{(s^2+9)^2}]_{s\to s-4}$

$= -6[\frac{[(s^2+9)^2(1) -(s)2(s^2+9)(2s)]}{(s^2+9)^4}]_{s\to s-4}$

$$= -6(s^2+9)\left[\frac{[(s^2+9)-4s^2]}{(s^2+9)^4}\right]_{s\to s-4}$$

$$= -6\left[\frac{9-3s^2}{(s^2+9)^3}\right]_{s\to s-4}$$

$$= \left[\frac{18s^2-54}{(s^2+9)^3}\right]_{s\to s-4}$$

$$\therefore \quad L[t^2e^{4t}sin3t] = \frac{18(s-4)^2-54}{((s-4)^2+9)^3}$$

Exercise: 5.2

Find the Laplace transform for the following

1. $tsinat$ **Ans:** $\frac{2as}{(s^2+a^2)^2}$
2. $tcosat$ **Ans**: $\frac{s^2-a^2}{(s^2+a^2)^2}$
3. $te^{-4t}sin3t$ **Ans:** $\frac{6(s+4)}{(s+4)^2+9}$
4. $tcos2tsin6t$ **Ans:** $\frac{8s}{(s^2+64)^2} - \frac{4s}{(s^2+16)^2}$
5. $te^{-2t}cos2t$ **Ans:** $\frac{(s-2)^2}{((s+4)^2+4)^2}$

Problems using the formula

$$\boldsymbol{L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]ds}$$

This formula is valid if $\lim_{t\to 0}\frac{f(t)}{t}$ is finite.

The following formula is very useful in this section

$$\int \frac{ds}{s} = logs$$

$$\int \frac{ds}{s+a} = \log(s+a)$$

$$\int \frac{s\,ds}{s^2+a^2} = \frac{1}{2}\log(s^2+a^2)$$

$$\int \frac{a\,ds}{s^2+a^2} = \tan^{-1}\frac{s}{a}$$

Example: 5.16 Find $\boldsymbol{L\left[\frac{cosat}{t}\right]}$

Solution:

$$\lim_{t\to 0}\frac{cosat}{t} = \frac{cosa(0)}{0} = \frac{1}{0} = \infty$$

∴ Laplace transform does not exists.

Example: 5.17 Find $\boldsymbol{L\left[\frac{sinat}{t}\right]}$

Solution:

$$\lim_{t\to 0}\frac{sinat}{t} = \frac{sina(0)}{0} = \frac{0}{0}$$

$$= \lim_{t\to 0} a\,cosat \qquad \text{(by applying L−Hospital rule)}$$

$$\lim_{t\to 0} a\ cosat = acos0 = a,$$ finite quantity.

Hence Laplace transform exists

$$L\left[\frac{sinat}{t}\right] = \int_s^\infty L[(sinat)]ds$$

$$= \int_s^\infty \frac{a}{s^2+a^2} ds$$

$$= \tan^{-1}\frac{s}{a}\Big]_s^\infty$$

$$= [\tan^{-1}\infty - \tan^{-1}\frac{s}{a}]$$

$$= [\frac{\pi}{2} - \tan^{-1}\frac{s}{a}]$$

$$\therefore L\left[\frac{sinat}{t}\right] = cot^{-1}\frac{s}{a}$$

Example: 5.18 Find $L\left[\frac{sin^3t}{t}\right]$

Solution:

$$\frac{sin^3t}{t} = \frac{3sint-sin3t}{4t}$$

$$\lim_{t\to 0}\frac{sin^3t}{t} = \lim_{t\to 0}\frac{3sint-sin3t}{4t}$$

$$= \frac{0-0}{0} = \frac{0}{0}$$ (by applying L−Hospital rule)

$$= \lim_{t\to 0}\frac{3sint-sin3t}{4t} = 0$$

Hence Laplace transform exists

$$L\left[\frac{sin^3t}{t}\right] = L\left[\frac{3sint-sin3t}{4t}\right]$$

$$= \frac{1}{4}\int_s^\infty L[(3sint - sin3t)]ds$$

$$= \frac{1}{4}\int_s^\infty \left(3\frac{1}{s^2+1} - \frac{3}{s^2+9}\right) ds$$

$$= \frac{1}{4}\left[3\tan^{-1}s - \tan^{-1}\frac{s}{3}\right]_s^\infty$$

$$= \frac{1}{4}[3(\tan^{-1}\infty - \tan^{-1}s) - (\tan^{-1}\infty - \tan^{-1}\frac{s}{3})]$$

$$= \frac{1}{4}[(\frac{\pi}{2} - \tan^{-1}s) - (\frac{\pi}{2} - \tan^{-1}\frac{s}{3})\ ]$$

$$= \frac{1}{4}[\cot^{-1}s - cot^{-1}\frac{s}{3}]$$

Example: 5.19 Find $L\left[e^{-2t}\frac{sin2tcos3t}{t}\right]$

Solution:

$$L\left[e^{-2t}\frac{sin2tcos3t}{t}\right] = L\left[\frac{sin2tcos3t}{t}\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty L(\sin(3t+2t) - \sin(3t-2t))ds\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty L((\sin 5t) - L(\sin t))ds\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty \left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}\right]\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left[\tan^{-1}\frac{s}{5} - \tan^{-1}s\right]_s^\infty\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left[\left(\tan^{-1}\infty - \tan^{-1}\frac{s}{5}\right) - \left(\tan^{-1}\infty - \tan^{-1}s\right)\right]\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \tan^{-1}\frac{s}{5}\right) - \left(\frac{\pi}{2} - \tan^{-1}s\right)\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\cot^{-1}\frac{s}{5} - cot^{-1}s\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\cot^{-1}\frac{(s+2)}{5} - cot^{-1}(s+2)\right]$$

Example: 5.20 Find the Laplace transform for $\frac{e^{-at}-e^{-bt}}{t}$

Solution:

$$\lim_{t\to 0}\frac{e^{-at}-e^{-bt}}{t} = \lim_{t\to 0}\frac{e^0-e^0}{0} = \frac{1-1}{0} = \frac{0}{0} \qquad \text{(use L− Hospital rule)}$$

$$= \lim_{t\to 0}\frac{-ae^{-at}+be^{-bt}}{1}$$

$$= -a + b = b - a = \text{a finite quantity}$$

Hence Laplace transform exists.

$$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty L[e^{-at} - e^{-bt}]ds$$

$$= \int_s^\infty [L(e^{-at}) - L(e^{-bt})]ds$$

$$= \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right]ds$$

$$= [\log(s+a) - \log(s+b)]_s^\infty$$

$$= \left[log\frac{s+a}{s+b}\right]_s^\infty$$

$$= \left[log\frac{s(1+\frac{a}{s})}{s(1+\frac{b}{s})}\right]_s^\infty$$

$$= log 1 - log\frac{s+a}{s+b} = 0 - log\frac{s+a}{s+b} \qquad \because log 1 = 0$$

$$= log\frac{s+a}{s+b}$$

Example: 5.21 Find the Laplace transform of $\frac{1-cost}{t}$

Solution:

$$\lim_{t\to 0}\frac{1-cost}{t} = \frac{0}{0} \qquad \lim_{t\to 0}\frac{sint}{1} = \frac{0}{1} = 0 \qquad \text{(use L− Hospital rule)}$$

$L\left[\frac{1-cost}{t}\right]$ exists.

$$L\left[\frac{1-cost}{t}\right] = \int_s^\infty L[(1-cost)]ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right)ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2 + 1)\right]_s^\infty$$

$$= \left[\log s - \log\sqrt{s^2 + 1}\right]_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2+1}}\right]_s^\infty$$

$$= 0 - \log \frac{s}{\sqrt{s^2+1}}$$

$$= \log \frac{\sqrt{s^2+1}}{s}$$

Example: 5.22 Find the Laplace transform for $\frac{cosat-cosbt}{t}$

Solution:

$$\lim_{t\to 0}\frac{cosat-cosbt}{t} = \frac{1-1}{0} = \frac{0}{0}$$ (use L− Hospital rule)

$$= \lim_{t\to 0}\frac{-asinat+bsinbt}{1} = 0 = \text{a finite quantity}$$

Hence Laplace transform exists.

$$L\left[\frac{cosat-cosbt}{t}\right] = \int_s^\infty L[cosat - cosbt]ds$$

$$= \int_s^\infty [L(cosat) - L(cosbt)]ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left[\frac{1}{2}\log(s^2 + a^2) - \frac{1}{2}\log(s^2 + b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{\left(1+\frac{a^2}{s^2}\right)}{\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log \frac{s^2+a^2}{s^2+b^2}\right] = -\frac{1}{2}\left[\log \frac{s^2+a^2}{s^2+b^2}\right] \qquad [\because \log 1 = 0]$$

$$= \frac{1}{2}\left[\log \frac{s^2+b^2}{s^2+a^2}\right]$$

Example: 5.23 Find the Laplace transform of $\frac{sin^2 t}{t}$

Solution:

$$\frac{sin^2 t}{t} = \frac{1-cos2t}{2t}$$

$$\lim_{t\to 0}\frac{\mathbf{1-cos2t}}{\mathbf{2t}} = \frac{0}{0}$$

$$\lim_{t\to 0}\frac{\mathbf{2sin2t}}{\mathbf{2}} = \frac{0}{1} = 0$$ (use L− Hospital rule)

Laplace transform exists.

$$L\left[\frac{\sin^2 t}{t}\right] = L\left[\frac{1-\cos 2t}{2t}\right] = \frac{1}{2}\int_s^\infty L[(1-\cos 2t)]ds$$

$$= \frac{1}{2}\int_s^\infty [L(1) - L(\cos 2t)]ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}[\log s - \log\sqrt{s^2+4}]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}\log\frac{\sqrt{s^2+4}}{s}$$

Example: 5.24 Find the Laplace transform for $\frac{\sin 2t \sin 5t}{t}$

Solution:

$$L\left[\frac{\sin 2t \sin 5t}{t}\right] = \int_s^\infty L[\sin 2t \sin 5t]ds$$

$$= \int_s^\infty \frac{1}{2}[L(\cos(-3t) - L(\cos 7t)]ds$$

$$= \frac{1}{2}\int_s^\infty [L(\cos(3t) - L(\cos 7t)]ds \qquad [\because \cos(-\theta) = \cos\theta]$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{s}{s^2+9} - \frac{s}{s^2+49}\right)ds$$

$$= \frac{1}{2}\left[\frac{1}{2}\log(s^2+9) - \frac{1}{2}\log(s^2+49)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\frac{s^2+9}{s^2+49}\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\frac{s^2\left(1+\frac{9}{s^2}\right)}{s^2\left(1+\frac{49}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\frac{\left(1+\frac{9}{s^2}\right)}{\left(1+\frac{49}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{4}\left[\log 1 - \log\frac{s^2+9}{s^2+49}\right] = -\frac{1}{4}\left[\log\frac{s^2+9}{s^2+49}\right] \qquad [\because \log 1 = 0]$$

$$= \frac{1}{4}\left[\log\frac{s^2+49}{s^2+9}\right]$$

Problems using $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$

Example: 5.25 Find the Laplace transform for (i) $\int_0^t e^{-2t}dt$ **(ii)** $\int_0^t \cos 2t\,dt$

(iii) $\int_0^t t\sin 3t\,dt$ **(iv)** $t\int_0^t \cos t\,dt$

Solution:

(i) $L\left[\int_0^t e^{-2t}dt\right] = \frac{1}{s}L[e^{-2t}] = \frac{1}{s}\left(\frac{1}{s+2}\right)$

$$\therefore L\left[\int_0^t e^{-2t}dt\right] = \frac{1}{s(s+2)}$$

(ii) $L\left[\int_0^t cos2tdt\right] = \frac{1}{s}L[cos2t] = \frac{1}{s}\left(\frac{s}{s^2+4}\right)$

$$\therefore L\left[\int_0^t cos2tdt\right] = \frac{1}{s^2+4}$$

(iii) $L\left[\int_0^t tsin3tdt\right] = \frac{1}{s}L[tsin3t]$

$$= \frac{1}{s}\left[\frac{-d}{ds}[L[sin3t]]\right]$$

$$= \frac{-1}{s}\left[\frac{d}{ds}\left[\frac{3}{s^2+9}\right]\right]$$

$$= \frac{-1}{s}\left[\frac{-6s}{(s^2+9)^2}\right]$$

$$\therefore L\left[\int_0^t tsin3tdt\right] = \frac{6}{(s^2+9)^2}$$

(iv) $L\left[t\int_0^t costdt\right] = \frac{-d}{ds}L\left[\int_0^t costdt\right]$

$$= \frac{-d}{ds}\left[\frac{1}{s}\left(\frac{s}{s^2+1}\right)\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$\therefore L\left[\int_0^t tsin3tdt\right] = \frac{2s}{(s^2+1)^2}$$

Example: 5.26 Find the Laplace transform for $e^{-t}\int_0^t tcos4tdt$

Solution:

$$L\left[e^{-t}\int_0^t tcos4tdt\right] = L\left[\int_0^t tcos4tdt\right]_{s\to s+1} = \left[\frac{-1}{s}\frac{d}{ds}L(cos4t)\right]_{s\to s+1}$$

$$= -\left(\frac{1}{s}\frac{d}{ds}\frac{s}{s^2+16}\right)_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{(s^2+16)1-s(2s)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{(s^2+16-2s^2)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{(-s^2+16)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\frac{(s^2-16)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$\therefore L\left[e^{-t}\int_0^t tcos4tdt\right] = \frac{1}{s+1}\left[\frac{(s+1)^2-16}{((s+1)^2+16)^2}\right]$$

Example: 5.27 Find the Laplace transform of $e^{-t}\int_0^t \frac{sint}{t}dt$

Solution:

$$L\left[e^{-t}\int_0^t \frac{sint}{t}dt\right] = L\left[\int_0^t \frac{sint}{t}dt\right]_{s\to s+1}$$

$$= \left[\frac{1}{s} L\left(\frac{\sin t}{t}\right)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\int_s^{\infty} L(\sin t)ds\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\int_s^{\infty} \frac{1}{s^2+1}\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}[\tan^{-1} s]_s^{\infty}\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}(\tan^{-1}\infty - \tan^{-1} s)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\left(\frac{\pi}{2} - \tan^{-1} s\right)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\cot^{-1} s\right]_{s\to s+1}$$

$$\therefore L\left[e^{-t}\int_0^t \frac{\sin t}{t}dt\right] = \frac{1}{s+1}\cot^{-1}(s+1)$$

Exercise: 5.3

Find the Laplace transform of

1. $\frac{\sin t}{t}$ **Ans:** $\cot^{-1}\frac{s}{2}$
2. $e^{-2t}\frac{\sin t}{t}$ **Ans:** $\cot^{-1}(s+2)$
3. $\frac{\sin at - \sin bt}{t}$ **Ans:** $\cot^{-1}\frac{s}{a} - \cot^{-1}\frac{s}{b}$
4. $\frac{e^{-at} - \cos bt}{t}$ **Ans:** $\log\frac{\sqrt{s^2+b^2}}{s+a}$
5. $\frac{1-e^{-t}}{t}$ **Ans:** $\log\frac{s+1}{s}$
6. $e^{-t}\int_0^t \frac{\sin t}{t}dt$ **Ans:** $\frac{1}{s+1}\cot^{-1}(s+1)$
7. $e^{-t}\int_0^t t\cos t\,dt$ **Ans:** $\frac{1}{s+1}\left[\frac{s^2+2s}{(s^2+2s+2)^2}\right]$
8. $e^{-t}\int_0^t te^{-t}\sin t\,dt$ **Ans:** $\frac{1}{s}\left[\frac{2(s+1)}{s^2+2s+2}\right]$

Evaluation of integrals using Laplace transform

Note: (i) $\int_0^{\infty} f(t)e^{-st}dt = L[f(t)]$

(ii) $\int_0^{\infty} f(t)e^{-at}dt = [L[f(t)]]_{s=a}$

(iii) $\int_0^{\infty} f(t)dt = [L[f(t)]]_{s=0}$

Example: 5.28 If $L[f(t)] = \frac{s+2}{s^2+4}$, then find the value of $\int_0^{\infty} f(t)dt$

Solution:

Given $L[f(t)] = \frac{s+2}{s^2+4}$

We know that $\int_0^{\infty} f(t)dt = [L[f(t)]]_{s=0}$

$$= \left[\frac{s+2}{s^2+4}\right]_{s=0} = \frac{2}{4}$$

$$\int_0^\infty f(t)dt = \frac{1}{2}$$

Example: 5.29 If $L[f(t)] = \frac{5s+4}{s^2-9}$, then find the value of $\int_0^\infty e^{-2t}f(t)dt$

Solution:

Given $L[f(t)] = \frac{5s+4}{s^2-9}$

We know that $\int_0^\infty e^{-2t}f(t)dt = [L[f(t)]]_{s=2}$

$$= \left[\frac{5s+4}{s^2-9}\right]_{s=2} = \frac{14}{-5}$$

$$\therefore \int_0^\infty e^{-2t}f(t)dt = \frac{-14}{5}$$

Example: 5.30 Find the values of the following integrals using Laplace transforms:

(i) $\int_0^\infty te^{-2t}cos2tdt$ **(ii)** $\int_0^\infty t^2e^{-t}sintdt$ **(iii)** $\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right)dt$

(iv) $\int_0^\infty \left(\frac{1-cost}{t}\right)e^{-t}dt$ **(v)** $\int_0^\infty \left(\frac{e^{-at}-cosbt}{t}\right)dt$

Solution:

(i) $\int_0^\infty te^{-2t}cos2tdt = L[tcos2t]_{s=2} = \left[\frac{-d}{ds}L(cos2t)\right]_{s=2}$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+4}\right)_{s=2}$$

$$= -\left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s=2}$$

$$= -\left[\frac{(4-s^2)}{(s^2+4)^2}\right]_{s=2}$$

$$= -\frac{(4-4)}{(4+4)^2} = 0$$

(ii) $\int_0^\infty t^2e^{-t}sintdt = L[t^2sint]_{s=1} = \frac{d^2}{ds^2}L[sint]_{s=1}$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right)_{s=1}$$

$$= \frac{d}{ds}\left[\frac{-1(2s)}{(s^2+1)^2}\right]_{s=1}$$

$$= -2\frac{d}{ds}\left[\frac{s}{(s^2+1)^2}\right]_{s=1}$$

$$= -2\left[\frac{\left[(s^2+1)^2(1)-s.2(s^2+1)(2s)\right]}{(s^2+1)^4}\right]_{s=1}$$

$$= -2\left[\frac{\left[(s^2+1)[(s^2+1)-4s^2]\right]}{(s^2+1)^4}\right]_{s=1}$$

$$= -2\left[\frac{(1-3s^2)}{(s^2+1)^3}\right]_{s=1}$$

$$= \left[\frac{6s^2-2}{(s^2+1)^3}\right]_{s=1} = \frac{4}{8} = \frac{1}{2}$$

(iii) $\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right)dt = L\left[\frac{e^{-t}-e^{-2t}}{t}\right]_{s=0} = \int_s^\infty [L[e^{-t}-e^{-2t}]ds]_{s=0}$

$$= \int_s^\infty [[L(e^{-t}) - L(e^{-2t})]ds] \quad s=0$$

$$= \int_s^\infty \left[\left(\frac{1}{s+1} - \frac{1}{s+2}\right) ds\right]_{s=0}$$

$$= \{[\log(s+1) - \log(s+2)]_s^\infty\}_{s=0}$$

$$= \left\{\left[log\frac{s+1}{s+2}\right]_s^\infty\right\}_{s=0}$$

$$= \left\{log\frac{s\left(1+\frac{1}{s}\right)}{s\left(1+\frac{2}{s}\right)}_s^\infty\right\}_{s=0}$$

$$= [0 - log\frac{s+1}{s+2}]_{s=0} \qquad \because log1 = 0$$

$$= [log\frac{s+2}{s+1}]_{s=0} = log2$$

(iv) $\int_0^\infty \left(\frac{1-cost}{t}\right) e^{-t} dt$

$$\int_0^\infty \left(\frac{1-cost}{t}\right) e^{-t} dt = L\left[\frac{1-cost}{t}\right]_{s=1} = \int_s^\infty [L[(1-cost)]ds]_{s=1}$$

$$= \int_s^\infty [[L(1) - L(cost)]ds]_{s=1}$$

$$= \int_s^\infty \left[\left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds\right]_{s=1}$$

$$= \left\{\left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty\right\}_{s=1}$$

$$= \{[\log s - \log\sqrt{s^2+1}]_s^\infty\}_{s=1}$$

$$= \{[log\frac{s}{\sqrt{s^2+1}}]_s^\infty\}_{s=1}$$

$$= [0 - log\frac{s}{\sqrt{s^2+1}}]_{s=1}$$

$$= [log\frac{\sqrt{s^2+1}}{s}]_{s=1}$$

$$= log\sqrt{2}$$

Exercise: 5.4

Find the values of the following integrals using Laplace transforms

1. $\int_0^\infty te^{-2t}costdt$ **Ans:** $\frac{3}{25}$
2. $\int_0^\infty te^{-3t}sintdt$ **Ans:** $\frac{13}{250}$
3. $\int_0^\infty \left(\frac{e^{-at}-e^{-bt}}{t}\right) dt$ **Ans:** $log\frac{b}{a}$
4. $\int_0^\infty e^{-2t}\frac{\sin^2 t}{t}dt$ **Ans:** $\frac{1}{4}log2$
5. $\int_0^\infty \left(\frac{cosat-cosbt}{t}\right) dt$ **Ans:** $log\frac{a}{b}$

Laplace transform of Piecewise continuous functions

$$\int_0^\infty f(t)e^{-st}dt = L[f(t)]$$

Example: 5.31 Find the Laplace transform of $f(t) = \begin{cases} e^{-t}; & 0 < t < \pi \\ 0\,; & t > \pi \end{cases}$

Solution:

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^\pi e^{-st}e^{-t}dt + \int_\pi^\infty e^{-st}0dt$$

$$= \int_0^\pi e^{-(s+1)t}dt$$

$$= \left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^\pi = \frac{e^{-(s+1)\pi}-e^0}{-(s+1)}$$

$$\therefore L[f(t)] = \frac{1-e^{-(s+1)\pi}}{-(s+1)}$$

Example: 5.32 Find the Laplace transform of $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0\,; & t > \pi \end{cases}$

Solution:

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^\pi e^{-st}\sin t\,dt + \int_\pi^\infty e^{-st}0dt$$

$$= \int_0^\pi e^{-st}\sin t\,dt$$

$$= \left[\frac{e^{-st}}{(-s)^2+1}(-s\sin t - \cos t)\right]_0^\pi = \frac{e^{-s\pi}}{s^2+1}[-s\sin\pi - \cos\pi] - \frac{e^0}{s^2+1}[-s\sin 0 - \cos 0]$$

$$= \frac{e^{-s\pi}}{s^2+1}(0+1) - \frac{1}{s^2+1}(-1) = \frac{e^{-s\pi}+1}{s^2+1}$$

$$\therefore L[f(t)] = \frac{e^{-s\pi}+1}{s^2+1} \qquad = 0 - \frac{e^{-sa}}{-s} = \frac{e^{-sa}}{s}$$

Exercise 5.5

1. Find the Laplace transform of $f(t) = \begin{cases} 0; 0 < t < 2 \\ 3\,; t > 2 \end{cases}$ **Ans:** $\frac{3e^{-2s}}{s}$
2. Find the Laplace transform of $f(t) = \begin{cases} e^t; 0 < t < 1 \\ 0\,; t > 1 \end{cases}$ **Ans:** $\frac{1-e^{-(s-1)}}{s-1}$
3. Find the Laplace transform of $f(t) = \begin{cases} 1; 0 < t < 1 \\ 0\,; t > 1 \end{cases}$ **Ans:** $\frac{1-e^{-s}}{s}$

Second Shifting theorem

Statement: If $L[f(t)] = F(s)$, then $L[f(t-a)U(t-a)] = e^{-as}F(s)$

Proof:

$$U(t-a)f(t-a) = \begin{cases} 0; & t < a \\ f(t-a)\,; & t > a \end{cases}$$

By the definition of Laplace transform,

$$L[U(t-a)f(t-a)] = \int_0^\infty U(t-a)f(t-a)e^{-st}dt$$

$$\int_0^a 0dt + \int_a^\infty f(t-a)e^{-st}dt$$

$$L[U(t-a)f(t-a)] = \int_0^\infty e^{-s(a+x)} f(x)dx$$
$$= \int_0^\infty e^{-sa} e^{-sx} f(x)dx$$
$$= e^{-sa} \int_0^\infty e^{-sx} f(x)dx$$

Let $t - a = x \cdots (1)$
$t = a + x$
$dt = dx$
When $t = a, (1) \Rightarrow x = 0$
When $t = \infty, (1) \Rightarrow x = \infty$

Replace x by t

$$L[U(t-a)f(t-a)] = e^{-sa} \int_0^\infty e^{-st} f(t)dt$$
$$= e^{-sa} L[f(t)] = e^{-sa} F(s)$$
$$L[U(t-a)f(t-a)] = e^{-sa} F(s)$$

5.5 Periodic functions

Definition: A function $f(t)$ is said to be periodic if $f(t + T) = f(t)$ for all values of t and for certain values of T. The smallest value of T for which $f(t + T) = f(t)$ for all t is called periodic function.

Example:

$sint = \sin(t + 2\pi) = \sin(t + 4\pi) \cdots$

$\therefore sint$ is periodic function with period 2π.

Let $f(t)$ be a periodic function with period T. Then

$$\boldsymbol{L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)dt}$$

Problems on Laplace transform of Periodic function

Example: 5.36 Find the Laplace transform of $f(t) = \begin{cases} E; & 0 \le t \le a \\ -E; & a \le t \le 2a \end{cases}$

Solution:

The given function is a periodic function with period $T = 2a$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$$
$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)dt$$
$$= \frac{1}{1-e^{-2as}} \left[\int_0^a Ee^{-st}dt + \int_a^{2a} -Ee^{-st}dt\right]$$
$$= \frac{1}{1-e^{-2as}} \left[E \int_0^a e^{-st}dt - E\int_a^{2a} e^{-st}dt\right]$$
$$= \frac{E}{1-e^{-2as}} \left[\left[\frac{e^{-st}}{-s}\right]_0^a - \left[\frac{e^{-st}}{-s}\right]_a^{2a}\right]$$
$$= \frac{E}{1-e^{-2as}} \left[\frac{e^{-as}}{-s} + \frac{1}{s} - \frac{e^{-2as}}{s} - \frac{e^{-as}}{s}\right]$$
$$= \frac{E}{1-e^{-2as}} \left[\frac{1-2e^{-as}+e^{-2as}}{s}\right]$$
$$= \frac{E}{1^2-(e^{-as})^2} \left[\frac{(1-e^{-as})^2}{s}\right]$$
$$= \frac{E}{(1-e^{-as})(1+e^{-as})} \left[\frac{(1-e^{-as})^2}{s}\right]$$

$$= \frac{E}{s}\frac{(1-e^{-as})}{(1+e^{-as})}$$

$$\therefore L[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{2}\right)$$

Example: 5.37 Find the Laplace transform of $f(t) = \begin{cases} 1; 0 \le t \le \frac{a}{2} \\ -1; \frac{a}{2} \le t \le a \end{cases}$ **given that** $f(t+a) = f(t)$.

Solution:

The given function is a periodic function with period $T = a$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}} (1)e^{-st}\,dt + \int_{\frac{a}{2}}^{a} (-1)e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}} e^{-st}\,dt - \int_{\frac{a}{2}}^{a} e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\left[\frac{e^{-st}}{-s}\right]_0^{\frac{a}{2}} - \left[\frac{e^{-st}}{-s}\right]_{\frac{a}{2}}^{2a}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\frac{e^{\frac{-sa}{2}}}{-s} + \frac{1}{s} + \frac{e^{-as}}{s} - \frac{e^{\frac{-sa}{2}}}{s}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\frac{1-2e^{\frac{-sa}{2}}+e^{-as}}{s}\right]$$

$$= \frac{1}{1^2-\left(e^{\frac{-sa}{2}}\right)^2}\left[\frac{\left(1-e^{\frac{-sa}{2}}\right)^2}{s}\right]$$

$$= \frac{1}{\left(1-e^{\frac{-sa}{2}}\right)\left(1+e^{\frac{-sa}{2}}\right)}\left[\frac{\left(1-e^{\frac{-sa}{2}}\right)^2}{s}\right]$$

$$= \frac{1}{s}\frac{\left(1-e^{\frac{-sa}{2}}\right)}{\left(1+e^{\frac{-sa}{2}}\right)} \qquad \left[\because \tanh x = \frac{(1-e^{-2x})}{(1+e^{-2x})}\right]$$

$$\therefore L[f(t)] = \frac{1}{s}\tanh\left(\frac{as}{4}\right)$$

Exercise 5.6

1. Find the Laplace transform of

$f(t) = \begin{cases} 1; 0 \le t \le \frac{a}{2} \\ -1; \frac{a}{2} \le t \le a \end{cases}$ given that $f(t+a) = f(t)$. **Ans:** $\frac{k}{s}\tanh\left(\frac{as}{2}\right)$

2. Find the Laplace transform of

$f(t) = \begin{cases} t; 0 \le t \le a \\ 2a - t; a \le t \le 2a \end{cases}$ given that $f(t+2a) = f(t)$. **Ans:** $\frac{1}{s^2}\tanh\left(\frac{s\pi}{2}\right)$

5.6 Inverse Laplace transform

Definition

If the Laplace transform of a function $f(t)$ is $F(s)$ *ie.*, $L[f(t)] = F(s)$, then $f(t)$ is called an inverse Laplace transform of $F(s)$ and we write symbolically $f(t) = L^{-1}[F(s)]$, where L^{-1} is called the inverse Laplace transform operator.

Inverse Laplace transform of elementary functions

$L[f(t)] = F(s)$	$L^{-1}[F(s)] = f(t)$
$L[1] = \frac{1}{s}$	$L^{-1}[\frac{1}{s}] = 1$
$L[t] = \frac{1}{s^2}$	$L^{-1}[\frac{1}{s^2}] = t$
$L[t^n] = \frac{n!}{s^{n+1}}$ $\quad$ *if n is an integer*	$L^{-1}[\frac{n!}{s^{n+1}}] = t^n$ $L^{-1}[\frac{1}{s^{n+1}}] = \frac{t^n}{n!}$
$L[e^{at}] = \frac{1}{s-a}$	$L^{-1}[\frac{1}{s-a}] = e^{at}$
$L[e^{-at}] = \frac{1}{s+a}$	$L^{-1}[\frac{1}{s+a}] = e^{-at}$
$L[sinat] = \frac{a}{s^2+a^2}$	$L^{-1}[\frac{1}{s^2+a^2}] = \frac{sinat}{a}$
$L[cosat] = \frac{s}{s^2+a^2}$	$L^{-1}[\frac{s}{s^2+a^2}] = cosat$
$L[sinhat] = \frac{a}{s^2-a^2}$	$L^{-1}[\frac{1}{s^2-a^2}] = \frac{sinhat}{a}$
$L[cosat] = \frac{s}{s^2-a^2}$	$L^{-1}[\frac{s}{s^2-a^2}] = coshat$

Result on inverse Laplace transform

Result: 1 Linear property

$L[f(t)] = F(s)$ **and** $L[g(t)] = G(s)$**, then** $L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]$

Where a and b are constants.

Proof:

We know that $L[aF(s) \pm bG(s)] = aL[F(s)] \pm bL[G(s)]$

$$= a\,F(s) \pm b\,G(s)$$

$(i.e.)\, a\,F(s) \pm b\,G(s) = L[af(t) \pm bg(t)]$

Operating L^{-1} on both sides, we get

$L^{-1}[aF(s) \pm bG(s)] = af(t) \pm bg(t)$

$L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]$

$\because f(t) = L^{-1}[F(s)]$
$\because g(t) = L^{-1}[G(s)]$

Result: 2 First shifting property

(i) $L^{-1}\,[F(s+a) = e^{-at}L^{-1}[F(s)]]$

(ii) $L^{-1}\,[F(s-a) = e^{at}L^{-1}[F(s)]]$

Proof:

Let $L[e^{-at}f(t)] = F[s+a]$

Operating L^{-1} on both sides, we get

$e^{-at}f(t) = L^{-1}[F[s+a]]$

$L^{-1}[F[s+a]] = e^{-at}L^{-1}[F(s)]$

Result: 3 Multiplication by s.

If $L^{-1}[F(s)] = f(t)$ **and** $f(0) = 0$**, then** $L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]$

Proof:

We know that $L[f'(t)] = sL[f(t)] - f(0) = sF(s)$

Operating L^{-1} on both sides, we get

$f'(t) = L^{-1}[sF(s)]$

$\frac{d}{dt}f(t) = L^{-1}[sF(s)]$

$\frac{d}{dt}L^{-1}[F(s)] = L^{-1}[sF(s)]$

$\therefore L^{-1}[sF(s)] = \frac{d}{ds}L^{-1}[F(s)]$

Result: 4 Division by s.

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]dt$$

Proof:

We know that $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)] = \frac{1}{s}F(s)$

Operating L^{-1} on both sides ,we get

$\int_0^t f(t)dt = L^{-1}\left[\frac{1}{s}F(s)\right]$

$\int_0^t L^{-1}[F(s)]\,dt = L^{-1}\left[\frac{1}{s}F(s)\right]$

$\therefore L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt$

Result: 5 Inverse Laplace transform of derivative

$$L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$$

Proof:

We know that $L[tf(t)] = \frac{-d}{ds}L[f(t)] = \frac{-d}{ds}F(s)$

Operating L^{-1} on both sides ,we get

$tf(t) = -L^{-1}\{\frac{d}{ds}F(s)\}$

$L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\{\frac{d}{ds}F(s)\}$

$f(t) = \frac{-1}{t}L^{-1}\{\frac{d}{ds}F(s)\}$

$L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\{\frac{d}{ds}F(s)\}$

Result: 6 Inverse Laplace transform of integral

$$L^{-1}[F(s)] = tL^{-1}\left[\int_s^\infty F(s)ds\right]$$

Proof:

We know that $L\left[\frac{f(t)}{t}\right] = \int_s^\infty L(f(t))\,ds$

$= \int_s^\infty \mathrm{F(s)ds}$

Operating L^{-1} on both sides, we get

$\frac{f(t)}{t} = L^{-1}\left[\int_s^\infty F(s)\,ds\right]$

$f(t) = tL^{-1}\left[\int_s^\infty F(s)\,ds\right]$

$L^{-1}[F(s)] = tL^{-1}\left[\int_s^\infty F(s)\,ds\right]$

Problems under inverse Laplace transform of elementary functions

Example: 5.39 Find the inverse Laplace for the following

(i) $\frac{1}{2s+3}$ **(ii)** $\frac{1}{4s^2+9}$ **(iii)** $\frac{s^3-3s^2+7}{s^4}$ **(iv)** $\frac{3s+5}{s^2+36}$

(i) $L^{-1}\left[\frac{1}{2s+3}\right] = L^{-1}\left[\frac{1}{2\left[s+\frac{3}{2}\right]}\right]$

$= \frac{1}{2}e^{\frac{-3t}{2}}$

(ii) $L^{-1}\left[\frac{1}{4s^2+9}\right] = L^{-1}\left[\frac{1}{4\left[s^2+\frac{9}{4}\right]}\right]$

$= \frac{1}{4}L^{-1}\left[\frac{1}{\left[s^2+\frac{9}{4}\right]}\right]$

$= \frac{1}{4}\frac{1}{3/2}\sin\frac{3}{2}t$

$= \frac{1}{6}\sin\frac{3}{2}t$

(iii) $L^{-1}\left[\frac{s^3-3s^2+7}{s^4}\right] = L^{-1}\left[\frac{s^3}{s^4} - \frac{3s^2}{s^4} + \frac{7}{s^4}\right]$

$= L^{-1}\left[\frac{1}{s}\right] - 3L^{-1}\left[\frac{1}{s^2}\right] + 7L^{-1}\left[\frac{1}{s^4}\right]$

$L^{-1}\left[\frac{s^3-3s^2+7}{s^4}\right] = 1 - 3t + \frac{7t^3}{3!}$

(iv) $L^{-1}\left[\frac{3s+5}{s^2+36}\right] = 3L^{-1}\left[\frac{s}{s^2+36}\right] + 5L^{-1}\left[\frac{1}{s^2+36}\right]$

$L^{-1}\left[\frac{3s+5}{s^2+36}\right] = 3\cos 6t + \frac{5\sin 6t}{6}$

Exercise: 5.7

Find the inverse Laplace transform for the following:

1. $\frac{2s-3}{s^2+5^2}$ **Ans:** $2\cos 5t - \frac{3\sin 5t}{5}$
2. $\frac{3s+5}{s^2+16}$ **Ans:** $3\cos 4t + \frac{5\sin 4t}{4}$
3. $\frac{1}{4s^2+9}$ **Ans:** $\frac{1}{6}\sin\frac{3}{2}t$
4. $\frac{1}{(s+4)^5}$ **Ans:** $e^{-4t}\frac{t^4}{4!}$
5. $\frac{1}{s^2-4s+13}$ **Ans:** $\frac{e^{2t}}{3}\sin 3t$

Inverse using the formula

$$L^{-1}[F(s)] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}F(s)\right]$$

Note: This formula is used when $F(s)$ is $\cot^{-1}\emptyset(s)$ or $\tan^{-1}\emptyset(s)$ or $log\emptyset(s)$

Example: 5.41 Find the inverse Laplace transform for the following

(i) $cot^{-1}\left(\frac{s}{a}\right)$ **(ii)** $tan^{-1}\left(\frac{a}{s}\right)$ **(iii)** $cot^{-1}as$

(iv) $tan^{-1}(s+a)$

Solution:

(i) $L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(cot^{-1}\left(\frac{s}{a}\right)\right)\right]$

$= \frac{-1}{t}L^{-1}\left[\frac{-1}{1+\frac{s^2}{a^2}}\left(\frac{1}{a}\right)\right] = \frac{1}{t}L^{-1}\left[\frac{-1}{\frac{a^2+s^2}{a^2}}\left(\frac{1}{a}\right)\right]$

$= \frac{1}{t}L^{-1}\left[\frac{a}{s^2+a^2}\right]$

$L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{1}{t}sinat$

(ii) $L^{-1}\left[tan^{-1}\left(\frac{a}{s}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(tan^{-1}\left(\frac{a}{s}\right)\right)\right]$

$= \frac{-1}{t}L^{-1}\left[\frac{1}{1+\left(\frac{a}{s}\right)^2}\left(\frac{-a}{s^2}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{1}{\frac{s^2+a^2}{s^2}}\left(\frac{-a}{s^2}\right)\right]$

$= \frac{1}{t}L^{-1}\left[\frac{a}{s^2+a^2}\right]$

$L^{-1}\left[tan^{-1}\left(\frac{a}{s}\right)\right] = \frac{1}{t}sinat$

(iii) $L^{-1}[cot^{-1}as] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}(cot^{-1}(as))\right]$

$= \frac{-1}{t}L^{-1}\left[\frac{-1}{1+a^2s^2}(a)\right] = \frac{1}{t}L^{-1}\left[\frac{a}{a^2\left(s^2+\frac{1}{a^2}\right)}\right]$

$= \frac{1}{at}L^{-1}\left[\frac{1}{s^2+\frac{1}{a^2}}\right] = \frac{1}{at}\left[\frac{sin\frac{1}{a}t}{\frac{1}{a}}\right]$

$L^{-1}[cot^{-1}as] = \frac{1}{t}sin\frac{t}{a}$

(iv) $L^{-1}[tan^{-1}(s+a)] = e^{-at}L^{-1}[tan^{-1}s]$

$= e^{-at}\left[\frac{-1}{t}L^{-1}\left[\frac{d}{ds}(tan^{-1}s)\right]\right]$

$= e^{-at}\left(\frac{-1}{t}\right)L^{-1}\left[\frac{1}{1+s^2}\right]$

$= \frac{-1}{t}e^{-at}L^{-1}\left[\frac{1}{1+s^2}\right]$

$L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{-e^{-at}}{t}sint$

Inverse using the formula

$$L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]$$

Example: 5.42 Find $L^{-1}\left[s\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$

Solution:

$$L^{-1}\left[s\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{d}{dt}L^{-1}\left[s\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] \cdots (1)$$

$$L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = L^{-1}\frac{d}{ds}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(\log(s^2+a^2) - \log(s^2+b^2)\right)\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{s^2+a^2}2s - \frac{1}{s^2+b^2}2s\right]$$

$$= \frac{-2}{t}L^{-1}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]$$

$$= \frac{-2}{t}[cosat - cosbt]$$

$$= \frac{2}{t}[cosbt - cosat]$$

Substituting in (1), we get

$$L^{-1}\left[s\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{d}{dt}\left[\frac{2}{t}[cosbt - cosat]\right]$$

$$= 2\left[\frac{t(-bsinbt+asinat)-(cosbt-cosat)}{t^2}\right]$$

$$L^{-1}\left[s\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = 2\left[\frac{t(-bsinbt+asinat)-(cosbt-cosat)}{t^2}\right]$$

Inverse using Partial Fraction

Example: 5.46 Find $L^{-1}\left[\frac{s-2}{s(s+2)(s-1)}\right]$

Solution:

$$\frac{s-2}{s(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$$

$$= \frac{A(s+2)(s-1)+Bs(s-1)+Cs(s+2)}{s(s+2)(s-1)}$$

$$A(s+2)(s-1) + Bs(s-1) + Cs(s+2) = s - 2 \cdots (1)$$

Put $s = 0$ in (1)	Put $s = -2$ in (1)	Put $s = 1$ in (1)
$A(2)(-1) = -2$	$B(-2)(-3) = -4$	$3C = -1$

$\Rightarrow A = 1$ $\qquad \Rightarrow B = \frac{-4}{6} = \frac{-2}{3}$ $\qquad \Rightarrow C = \frac{-1}{3}$

$$\therefore \frac{s-2}{s(s+2)(s-1)} = \frac{1}{s} - \frac{2}{s+2} - \frac{1}{3(s-1)}$$

$$L^{-1}\left[\frac{s-2}{s(s+2)(s-1)}\right] = 1 - \frac{2}{3}e^{-2t} - \frac{1}{3}e^{t}$$

Example: 5.47 Find $L^{-1}\left[\frac{2s-3}{(s-1)(s-2)^2}\right]$

Solution:

$$\frac{2s-3}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$= \frac{A(s-2)^2 + B(s-1)(s-2) + C(s-1)}{(s-1)(s-2)^2}$$

$$A(s-2)^2 + B(s-1)(s-2) + C(s-1) = 2s - 3 \cdots (1)$$

Put $s = 1$ in (1)	Put $s = 2$ in (1)	Equating the coefficient of s^2
		$A + B = 0$
$A = -1$	$C = 1$	$B = -A \Rightarrow B = 1$

$$\therefore \frac{2s-3}{(s-1)(s-2)^2} = \frac{-1}{s-1} + \frac{1}{s-2} + \frac{1}{(s-2)^2}$$

$$\therefore L^{-1}\left[\frac{2s-3}{(s-1)(s-2)^2}\right] = -e^{t} + e^{2t} + e^{2}\ t$$

Example: 5.48 Find the inverse Laplace transform of $\frac{5s^2-15s-11}{(s+1)(s-2)^3}$

Solution:

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{C}{(s-2)^3}$$

$$= \frac{A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)}{(s-1)(s-2)^3}$$

$$A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1) = 5s^2 - 15s - 11 \cdots (1)$$

Put $s = -1$ in (1)	Put $s = 2$ in (1)	Equating the coefficient of s^3
$A(-27) = 9$	$D(3) = -21$	$A + B = 0$
$A = \frac{9}{-27} \Rightarrow A = \frac{-1}{3}$	$D = \frac{-21}{3} = -7$	$B = -A \Rightarrow B = \frac{1}{3}$

Put $s = 0$ in (1), we get

$-8A + 4B - 2C + D = -11$

$$\frac{8}{3}+\frac{4}{3}-2C-7=-11$$

$$4-2C=7-11$$

$$-2C=-8 \Rightarrow C=4$$

$$\therefore \frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{-1}{3(s+1)}+\frac{1}{3(s-2)}+\frac{4}{(s-2)^2}-\frac{7}{(s-2)^3}$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]=\frac{-1}{3}L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{3}L^{-1}\left[\frac{1}{s-2}\right]+4L^{-1}\left[\frac{1}{(s-2)^2}\right]-7L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$=\frac{-1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]-7e^{2t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]=\frac{-1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]-7e^{2t}\frac{t^2}{2}$$

Exercise 5.8

Find the Inverse Laplace transforms using partial fraction for the following

1. $\frac{1}{(s+1)(s+3)}$ **Ans:** $\frac{1}{2}(e^{-t}-e^{-3t})$

2. $\frac{1}{s(s+1)(s+2)}$ **Ans:** $\frac{1}{2}(e^{-2t}-2e^{-t}+1)$

5.7 Convolution theorem

Definition: Convolution of two functions

The convolution of two functions $f(t)$ and $g(t)$ is denoted by $f(t)*g(t)$ and defined by

$f(t)*g(t)=\int_0^t f(u)g(t-u)du.$

State and prove Convolution theorem

Statement: If $L[f(t)]=F(s)$ and $L[g(t)]=G(s)$, then $L[f(t)]*L[g(t)]=F(s)G(s)$

Proof:

We have $f(t) * g(t) = \int_0^t f(u)g(t-u)du$

$L[f(t) * g(t)] = \int_0^\infty [f(t) * g(t)]\, e^{-st}dt$

$= \int_0^\infty \int_0^t f(u)g(t-u)du e^{-st}dt$

$= \int_0^\infty \int_0^t f(u)g(t-u)e^{-st}dudt \cdots (1)$

Now we have no change the order of integration.

$u = 0, u = t; t = 0, t = \infty$

Change of order is . Draw horizontal strip PQ

At P, $t = u$, At A $u = \infty$

$L[f(t) * g(t)] = \int_0^\infty \int_u^\infty f(u)g(t-u)e^{-st}dtdu$

$= \int_0^\infty f(u)\, [\int_u^\infty g(t-u)e^{-st}dt]du \cdots (2)$

Put $t - u = x \cdots (3)$

$t = u + x \Rightarrow dt = dx$

When $t = u; (3) \Rightarrow x = 0$

When $t = \infty; (3) \Rightarrow x = \infty$

(1) $\Rightarrow L[f(t) * g(t)] = \int_0^\infty f(u)\, [\int_0^\infty g(x)e^{-s(u+x)}dx]du$

$= \int_0^\infty f(u)\, [\int_0^\infty g(x)e^{-su}e^{-sx}dx]du$

$= \int_0^\infty f(u)e^{-su}du \int_0^\infty g(x)e^{-sx}dx$

$= L[f(u)]L[g(x)]$

$\therefore L[f(t) * g(t)] = F(s)G(s)$

Note: Convolution theorem is very useful to compute inverse Laplace transform of product of two terms

Convolution theorem is $L[f(t) * g(t)] = F(s)G(s)$

$L^{-1}[F(s)G(s)] = f(t) * g(t)$

$L^{-1}[F(s)G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$

Problems under Convolution theorem

Example: 5.56 Find the inverse Laplace transform $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.

Solution:

$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}\ \frac{s}{(s^2+b^2)}\right]$

$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{s}{(s^2+b^2)}\right]$

$= cosat * cosbt$

$= \int_0^t cosau\, cosb(t-u)du$

$$= \int_0^t \frac{\cos(au+bt-bu)+\cos(au-bt+bu)}{2} du$$

$$= \frac{1}{2}\int_0^t (\cos(au + bt - bu) + \cos(au - bt + bu))\, du$$

$$= \frac{1}{2}\int_0^t [\cos(a-b)\,u + bt + \cos(a+b)\,u - bt] du$$

$$= \frac{1}{2}\left[\frac{sin[(a-b)u+bt]}{a-b} + \frac{sin[(a+b)u+bt]}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(at-bt+bt)}{a-b} + \frac{\sin(at-bt+bt)}{a+b} - \frac{sinbt}{a-b} + \frac{sinbt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{sinbt}{a-b} + \frac{sinbt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{(a+b)sinat+(a-b)sinat-(a+b)sinbt+(a-b)sinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2asinat-2bsinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2(asinat-bsinbt)}{a^2-b^2}\right]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{asinat-bsinbt}{a^2-b^2}$$

Example: 5.58 Find the inverse Laplace transform $\frac{s}{(s^2+4)(s^2+9)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] = L^{-1}\left[\frac{1}{(s^2+4)}\,\frac{s}{(s^2+9)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+4)}\right] * L^{-1}\left[\frac{s}{(s^2+9)}\right]$$

$$= \frac{1}{2} sin2t * cos3t$$

$$= \frac{1}{2}\int_0^t sin2u\, cos3(t-u) du$$

$$= \frac{1}{2}\int_0^t \frac{\sin(2u+3t-3u)+\sin(2u-3t+3u)}{2} du$$

$$= \frac{1}{4}\int_0^t [\sin(3t - u) + \sin(5u - 3t)] du$$

$$= \frac{1}{4}\left[\frac{-\cos(3t-u)}{-1} - \frac{\cos(5u-3t)}{5}\right]_0^t$$

$$= \frac{1}{4}\left[\frac{\cos(3t-t)}{1} - \frac{\cos(5t-3t)}{5} - \frac{cos3t}{1} + \frac{cos3t}{5}\right]$$

$$= \frac{1}{4}\left[cos2t - \frac{\cos 2t}{5} - cos3t + \frac{cos3t}{5}\right]$$

$$= \frac{1}{4}\left[\frac{5cos2t-cos2t-5cos3t+cos3t}{5}\right]$$

$$= \frac{1}{20}[4cos2t - 4cos3t]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] = \frac{cos2t-cos3t}{5}$$

Exercise: 5.10

Find the inverse Laplace transforms using convolution theorem for the following

1. $\frac{1}{s(s^2+1)}$ **Ans:** $1 - cost$
2. $\frac{s}{(s^2+4)^2}$ **Ans:** $\frac{1}{8}\left[\frac{\sin 2t}{2} - tcos2t\right]$
3. $\frac{s^2}{(s^2+4)^2}$ **Ans:** $\frac{1}{2}\left[tcos2t + \frac{\sin 2t}{2}\right]$
4. $\frac{1}{(s+1)(s^2+1)}$ **Ans:** $\frac{1}{2}\left[e^{-t} + sint - cost\right]$
5. $\frac{1}{(s+1)(s^2+4)}$ **Ans:** $-\frac{1}{5}e^{-t} + \frac{1}{5}cos2t - \frac{1}{10}sin2t$

5.8 Solution of differential equation by Laplace transform technique

There are so many methods to solve a linear differential equation. If the initial conditions are known, then Laplace transform technique is easier to solve the differential equation. The Laplace transform transforms the differential equation into an algebraic equation.

$$L[y'(t)] = sL[y(t)] - y(0)$$

$$L[y''(t)] = s^2L[y(t)] - sy(0) - y'(0)$$

Problems using Partial Fraction

Example: 5.66 Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2$, given $x = 0$ and $\frac{dx}{dt} = 5$ for $t = 0$ using Laplace transform method.

Solution:

Given $x'' - 3x' + 2x = 2; x(0) = 0; x'(0) = 5$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] - 3L[x'(t)] + 2L[x(t)] = 2L(1)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] - 3[sL[x(t)] - x(0)] + 2L[x(t)] = \frac{2}{s}$$

Substituting $x(0) = 0; x'(0) = 5$

$$[s^2L[x(t)] - 0 - 5] - 3[sL[x(t)] - 0] + 2L[x(t)] = \frac{2}{s}$$

$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$

$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$

Put $L[x(t)] = \bar{x}$

$$s^2\bar{x} - 3s\bar{x} + 2\bar{x} = \frac{2}{s} + 5$$

$$[s^2 - 3s + 2]\bar{x} = \frac{2}{s} + 5$$

$$(s-1)(s-2)\bar{x} = \frac{2}{s} + 5$$

$$\bar{x} = \frac{2+5s}{s(s-1)(s-2)}$$

Consider $\frac{2+5s}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{A(s-1)(s-2)+Bs(s-2)+Cs(s-1)}{s(s-1)(s-2)}$$

$$A(s-1)(s-2) + Bs(s-2) + Cs(s-1) = 2 + 5s \cdots (1)$$

Put $s = 0$ in (1)	Put $s = 1$ in (1)	Put $s = 2$ in (1)
$A(-1)(-2) = 2$	$B(1)(-1) = 7$	$C(2)(1) = 2 + 10$
$A = 1$	$B = -7$	$C = 6$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{1}{s} - \frac{7}{s-1} + \frac{6}{s-2}$$

$\therefore \bar{x} = \frac{1}{s} - 7\frac{1}{s-1} + 6\frac{1}{s-2}$

$x(t) = 1 - 7e^{t} + 6e^{2t}$

Example: 5.67 Using Laplace transform solve the differential equation $y'' - 3y' - 4y = 2e^{-t}$, with $y(0) = 1 = y'(0)$.

Solution:

Given $y'' - 3y' - 4y = 2e^{-t}$; with $y(0) = 1 = y'(0)$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 3L[y'(t)] - 4L[y(t)] = 2L(e^{-t})$

$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] - 4L[y(t)] = 2\frac{1}{s+1}$

Substituting $y(0) = 1 = y'(0)$.

$[s^2L[y(t)] - s - 1] - 3[sL[y(t)] - 1] - 4L[y(t)] = \frac{2}{s+1}$

$s^2L[y(t)] - s - 1 - 3sL[y(t)] + 3 - 4L[y(t)] = \frac{2}{s+1}$

$s^2L[y(t)] - 3sL[y(t)] - 4L[y(t)] = \frac{2}{s+1} + s - 2$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 3s\bar{y} - 4\bar{y} = \frac{2}{s+1} + s - 2$

$[s^2 - 3s - 4]\bar{y} = \frac{2}{s+1} + s - 2$

$[s^2 - 3s - 4]\bar{y} = \frac{2+s(s+1)-2(s+1)}{s+1}$

$= \frac{2+s^2+s-2s-2}{s+1}$

$(s+1)(s-4)\bar{y} = \frac{s^2-s}{s+1}$

$\bar{y} = \frac{s^2-s}{(s+1)(s+1)(s-4)}$

$\bar{y} = \frac{s^2-s}{(s+1)^2(s-4)}$

Consider $\frac{s^2-s}{(s+1)^2(s-4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-4}$

$\frac{s^2-s}{(s+1)^2(s-4)} = \frac{A(s+1)(s-4)+B(s-4)+C(s+1)^2}{(s+1)^2(s-4)}$

$A(s+1)(s-4) + B(s-4) + C(s+1)^2 = s^2 - s \cdots (1)$

Put $s = -1$ in (1)	Put $s = 4$ in (1)	equating the coefficients of s^2, we get
$-5B = 1 + 1$	$25C = 16 - 4$	$A + C = 1 \Rightarrow A = 1 - C \Rightarrow 1 - \frac{12}{25}$
$B = \frac{-2}{5}$	$C = \frac{12}{25}$	$A = \frac{13}{25}$

$\frac{s^2-s}{(s+1)^2(s-4)} = \frac{25}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$

$\therefore \bar{y} = \frac{13}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$

$y(t) = \frac{13}{25} L^{-1}\left[\frac{1}{(s+1)}\right] - \frac{2}{5} L^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{12}{25} L^{-1}\left[\frac{1}{s-4}\right]$

$y(t) = \frac{13}{25} e^{-t} - \frac{2}{5} te^{-t} + \frac{12}{25} e^{4t}$

Example: 5.68 Solve the differential equation $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{-t}$, with $y(0) = 1$ and $y'(0) = 0$ using Laplace transform.

Solution:

Given $y'' - 3y' + 2y = e^{-t}$; with $y(0) = 1$ and $y'(0) = 1$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L(e^{-t})$

$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = \frac{1}{s+1}$

Substituting $y(0) = 1$ and $y'(0) = 0$.

$[s^2L[y(t)] - s - 0] - 3[sL[y(t)] - 1] + 2L[y(t)] = \frac{1}{s+1}$

$s^2L[y(t)] - s - 3sL[y(t)] + 3 + 2L[y(t)] = \frac{1}{s+1}$

$s^2L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{1}{s+1} + s - 3$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{1}{s+1} + s - 3$

$[s^2 - 3s + 2]\bar{y} = \frac{1}{s+1} + s - 3$

$[s^2 - 3s + 2]\bar{y} = \frac{1 + s(s+1) - 3(s+1)}{s+1}$

$= \frac{1 + s^2 + s - 3s - 3}{s+1}$

$(s-1)(s-2)\bar{y} = \frac{s^2 - 2s - 2}{s+1}$

$\bar{y} = \frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)}$

Consider $\frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s-2}$

$\frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \frac{A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1)}{(s+1)(s-1)(s-2)}$

$A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1) = s^2 - 2s - 2 \cdots (1)$

Put $s = -1$ in (1)	put $s = 1$ in (1)	put $s = 2$ in (1)
$6A = 1 + 2 - 2$	$-2B = 1 - 4$	$3C = 4 - 4 - 2$
$A = \frac{1}{6}$	$B = \frac{3}{2}$	$C = \frac{-2}{3}$

$\therefore \frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$

$\bar{y} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$

$$y(t) = \frac{1}{6}L^{-1}\left[\frac{1}{(s+1)}\right] + \frac{3}{2}L^{-1}\left[\frac{1}{s-1}\right] - \frac{2}{3}L^{-1}\left[\frac{1}{s-2}\right]$$

$$y(t) = \frac{1}{6}e^{-t} + \frac{3}{2}e^{t} - \frac{2}{3}e^{2t}$$

Example: 5.69 Using Laplace transform solve the differential equation $y'' + 2y' - 3y = sint$, with $y(0) = y'(0) = 0$.

Solution:

Given $y'' + 2y' - 3y = sint$ with $y(0) = 0 = y'(0)$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] + 2L[y'(t)] - 3L[y(t)] = L(sint)$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + 2[sL[y(t)] - y(0)] - 3L[y(t)] = \frac{1}{s^2+1}$$

Substituting $y(0) = 0 = y'(0)$.

$$[s^2L[y(t)] - 0 - 0] + 2[sL[y(t)] - 0] - 3L[y(t)] = \frac{1}{s^2+1}$$

$$s^2L[y(t)] + 2sL[y(t)] - 3L[y(t)] = \frac{1}{s^2+1}$$

$$s^2L[y(t)] + 2sL[y(t)] - 3L[y(t)] = \frac{1}{s^2+1}$$

Put $L[y(t)] = \bar{y}$

$$s^2\bar{y} + 2s\bar{y} - 3\bar{y} = \frac{1}{s^2+1}$$

$$[s^2 + 2s - 3]\bar{y} = \frac{1}{s^2+1}$$

$$(s-1)(s+3)\bar{y} = \frac{1}{s^2+1}$$

$$\bar{y} = \frac{1}{(s-1)(s+3)(s^2+1)}$$

Consider $\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A(s^2+1)(s+3) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3)}{(s-1)(s+3)(s^2+1)}$$

$$A(s^2+1)(s+3) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3) = 1 \cdots (1)$$

Put $s = 1$ in (1)	Put $s = -3$ in (1)	equating the coefficients of s^2, we get
$8A = 0 + 1$	$B(-4)(10) = 1$	$A + B + C = 0 \Rightarrow C = -A - B = \frac{-1}{8} + \frac{1}{40}$
$A = \frac{1}{8}$	$B = \frac{-1}{40}$	$C = \frac{-1}{10}$

Put $s = 0$ in (1), we get

$$3A - B - 3D = 1 \Rightarrow \frac{3}{8} + \frac{1}{40} - 3D = 1$$

$$3D = \frac{3}{8} + \frac{1}{40} - 1$$

$$3D = \frac{15+1-40}{40} \Rightarrow D = \frac{-24}{40\times3} \Rightarrow D = \frac{-1}{5}$$

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{1}{8(s-1)} - \frac{1}{40(s+3)} + \frac{(\frac{-1}{10})s - \frac{1}{5}}{s^2+1}$$

$\therefore \bar{y} = \frac{1}{8(s-1)} - \frac{1}{40(s+3)} - \frac{s}{10(s^2+1)} - \frac{1}{5(s^2+1)}$

$y(t) = \frac{1}{8} L^{-1}\left[\frac{1}{(s-1)}\right] - \frac{1}{40} L^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10} L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{5} L^{-1}\left[\frac{1}{s^2+1}\right]$

$y(t) = \frac{1}{8} e^{t} - \frac{1}{40} e^{-3t} - \frac{1}{10}(cost - 2sint)$

Example: 5.70 Using Laplace transform solve the differential equation $y'' - 3y' + 2y = 4e^{2t}$, with $y(0) = -3$ and $y'(0) = 5$.

Solution:

Given $y'' - 3y' + 2y = 4e^{2t}$; with $y(0) = -3$ and $y'(0) = 5$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = 4L(e^{2t})$

$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = 4\frac{1}{s-2}$

Substituting $y(0) = -3$ and $y'(0) = 5$.

$[s^2L[y(t)] + 3s - 5] - 3[sL[y(t)] + 3] + 2L[y(t)] = \frac{4}{s-2}$

$s^2L[y(t)] + 3s - 5 - 3sL[y(t)] - 9 + 2L[y(t)] = \frac{4}{s-2}$

$s^2L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{4}{s-2} - 3s + 14$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{4}{s-2} - 3s + 14$

$[s^2 - 3s + 2]\bar{y} = \frac{4}{s-2} + 14 - 3s$

$[s^2 - 3s + 2]\bar{y} = \frac{4+(14-3s)(s-2)}{s-2}$

$(s-1)(s-2)\bar{y} = \frac{4+(14-3s)(s-2)}{s-2}$

$\bar{y} = \frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2}$

Consider $\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$

$\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A(s-2)^2 + B(s-1)(s-2) + C(s-1)}{(s-1)(s-2)^2}$

$A(s-2)^2 + B(s-1)(s-2) + C(s-1) = 4 + (14-3s)(s-2) \cdots (1)$

Put $s = 1$ in (1)	Put $s = 2$ in (1)	equating the coefficients of s^2, we get
$A = 4 - 11$	$C = 4 + 0$	$A + B = -3 \Rightarrow -7 + B = -3$
$A = -7$	$C = 4$	$B = 4$

$\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

$\therefore \bar{y} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

$y(t) = -7e^{t} + 4e^{2t} + 4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$

$y(t) = -7e^{t} + 4e^{2t} + 4e^{2t}t$

Example: 5.71 Using Laplace transform solve the differential equation $y'' - 4y' + 8y = e^{2t}$, with $y(0) = 2$ and $y'(0) = -2$.

Solution:

Given $y'' - 4y' + 8y = e^{2t}$; with $y(0) = 2$ and $y'(0) = -2$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 4L[y'(t)] + 8L[y(t)] = L(e^{2t})$

$[s^2L[y(t)] - sy(0) - y'(0)] - 4[sL[y(t)] - y(0)] + 8L[y(t)] = \frac{1}{s-2}$

Substituting $y(0) = 2$ and $y'(0) = -2$.

$[s^2L[y(t)] - 2s + 2] - 4[sL[y(t)] - 2] + 8L[y(t)] = \frac{1}{s-2}$

$s^2L[y(t)] - 2s + 2 - 4sL[y(t)] + 8 + 8L[y(t)] = \frac{1}{s-2}$

$s^2L[y(t)] - 4sL[y(t)] + 8L[y(t)] = \frac{1}{s-2} + 2s - 10$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 4s\bar{y} + 8\bar{y} = \frac{1}{s-2} + 2s - 10$

$[s^2 - 4s + 8]\bar{y} = \frac{1}{s-2} + 2s - 10$

$[s^2 - 4s + 8]\bar{y} = \frac{1+(2s-10)(s-2)}{s-2}$

$\bar{y} = \frac{1+(2s-10)(s-2)}{(s-2)(s^2-4s+8)}$

$= \frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]}$

Consider $\frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{A}{s-2} + \frac{B(s-2)+C}{(s-2)^2+4}$

$= \frac{A[(s-2)^2+4]+B[(s-2)+C](s-2)}{[s-2][(s-2)^2+4]}$

$A[(s-2)^2 + 4] + B[(s-2) + C](s-2) = 1 + (2s-10)(s-2) \cdots (1)$

Put $s = 2$ in (1)

$4A = 1 + 0$

$A = \frac{1}{4}$

Put $s = 0$ in (1)

$8A + 4B - 2C = 21$

$C = -6$

equating the coefficients of s^2, we get

$A + B = 2 \Rightarrow \frac{1}{4} + B = 2$

$B = \frac{7}{4}$

$\frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{\frac{1}{4}}{s-2} + \frac{\frac{7}{4}(s-2)-6}{(s-2)^2+4}$

$\therefore \bar{y} = \frac{1}{4(s-2)} + \frac{7}{4}\frac{(s-2)}{(s-2)^2+4} - 6\frac{1}{(s-2)^2+4}$

$y(t) = \frac{1}{4}L^{-1}\left[\frac{1}{(s-2)}\right] + \frac{7}{4}L^{-1}\left[\frac{(s-2)}{(s-2)^2+4}\right] - 6L^{-1}\left[\frac{1}{(s-2)^2+4}\right]$

$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}L^{-1}\left[\frac{s}{s^2+4}\right] - 6e^{2t}L^{-1}\left[\frac{1}{s^2+4}\right]$

$= \frac{1}{4} e^{2t} + \frac{7}{4} e^{2t} cos2t - 6e^{2t} \frac{sin2t}{2}$

$y(t) = \frac{1}{4} e^{2t} + \frac{7}{4} e^{2t} cos2t - 3e^{2t} sin2t$

Problems without using Partial Fraction

Example: 5.72 Solve using Laplace transform $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$, with $x = 2, \frac{dx}{dt} = -1$ at $t = 0$

Solution:

Given $x'' - 2x' + x = e^t; x(0) = 2; x'(0) = -1$

Taking Laplace transform on both sides, we get,

$L[x''(t)] - 2L[x'(t)] + L[x(t)] = L(e^t)$

$[s^2L[x(t)] - sx(0) - x'(0)] - 2[sL[x(t)] - x(0)] + L[x(t)] = \frac{1}{s-1}$

Substituting $x(0) = 2; x'(0) = -1$

$[s^2L[x(t)] - 2s + 1] - 2[sL[x(t)] - 2] + L[x(t)] = \frac{1}{s-1}$

$s^2L[x(t)] - 2sL[x(t)] + L[x(t)] = \frac{1}{s-1} + 2s - 5$

$s^2L[x(t)] - 2sL[x(t)] + L[x(t)] = \frac{1}{s-1} + 2s - 5$

Put $L[x(t)] = \bar{x}$

$s^2\bar{x} - 2s\bar{x} + \bar{x} = \frac{1}{s-1} + 2s - 5$

$[s^2 - 2s + 1]\bar{x} = \frac{1}{s-1} + 2s - 5$

$(s-1)^2\bar{x} = \frac{1}{s-1} + 2s - 5$

$x(t) = L^{-1}\left[\frac{1}{(s-1)^3}\right] + 2L^{-1}\left[\frac{s}{(s-1)^2}\right] - 5L^{-1}\left[\frac{1}{(s-1)^2}\right]$

$= e^t L^{-1}\left[\frac{1}{s^3}\right] + 2L^{-1}\left[\frac{s-1+1}{(s-1)^2}\right] - 5e^t L^{-1}\left[\frac{1}{s^2}\right]$

$= e^t \frac{t^2}{2!} + 2L^{-1}\left[\frac{s-1}{(s-1)^2} + \frac{1}{(s-1)^2}\right] - 5e^t t$

$= e^t \frac{t^2}{2!} + 2L^{-1}\left[\frac{1}{s-1}\right] + 2L^{-1}\left[\frac{1}{(s-1)^2}\right] - 5e^t t$

$= e^t \frac{t^2}{2!} + 2e^t + 2e^t L^{-1}\left[\frac{1}{s^2}\right] - 5e^t t$

$= e^t \frac{t^2}{2} + 2e^t + 2e^t t - 5e^t t$

$\therefore x = \frac{t^2 e^t}{2} + 2e^t - 3e^t t$

Exercise: 5.11

1. Solve using Laplace transform $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 5$, given that $= 0$,$\frac{dy}{dt} = 2$ when $t = 0$

Ans:$-1 - \frac{1}{6}e^{-5t} + \frac{5}{6}e^{t}$

2. Using Laplace transform solve the differential equation $y'' + 5y' + 6y = 2$, with $y(0) = 0 = y'(0)$. Where $y' = \frac{dy}{dt}$ **Ans:**$y(t) = \frac{1}{3} - e^{-2t} + \frac{2}{3}e^{-3t}$

3. Using Laplace transform solve the differential equation $y'' + 4y' + 3y = e^{-t}$, with $y(0) = 1;\ y'(0) = 0.$ **Ans:**$y(t) = \frac{-1}{4}e^{-3t} - \frac{5}{4}e^{-t} + \frac{1}{2}te^{-t}$

4. Solve using Laplace transform $\frac{d^2y}{dt^2} + y = sint$ given $= 1$,$\frac{dy}{dt} = 0$ when $t = 0$

Ans:$y(t) = sint - tcost$

5. Solve using Laplace transform$\frac{d^2y}{dt^2} + 9y = cos2t$, if $y(0) = 1$;$y\left(\frac{\pi}{2}\right) = -1$

Ans:$y(t) = \frac{1}{5}[cos2t + 4cos3t + 4sin3t]$

REFERENCES

1. Abramowitz, M. and I. A. Stegun (eds.), Handbook of Mathematical Functions. 10th printing, with corrections. Washington, DC: National Bureau of Standards. 1972 (also New York: Dover, 1965).
2. Cajori, F., History of Mathematics. 5th ed. Reprinted. Providence, RI: American Mathematical Society, 2002.
3. Courant, R. and D. Hilbert, Methods of Mathematical Physics. 2 vols. Hoboken, NJ: Wiley, 1989.
4. Courant, R., Differential and Integral Calculus. 2 vols. Hoboken, NJ: Wiley, 1988.
5. Graham, R. L. et al., Concrete Mathematics. 2nd ed. Reading, MA: Addison-Wesley, 1994.
6. Ito, K. (ed.), Encyclopedic Dictionary of Mathematics. 4 vols. 2nd ed. Cambridge, MA: MIT Press, 1993.
7. Kreyszig, E., Introductory Functional Analysis with Applications. New York: Wiley, 1989.
8. Kreyszig, E., Differential Geometry. Mineola, NY: Dover, 1991.
9. Kreyszig, E. Introduction to Differential Geometry and Riemannian Geometry. Toronto: University of Toronto Press, 1975.
10. Szegö, G., Orthogonal Polynomials. 4th ed. Reprinted. New York: American Mathematical Society, 2003.
11. Thomas, G. et al., Thomas' Calculus, Early Transcendentals Update. 10th ed. Reading, MA: Addison-Wesley, 2003.
12. Arnold, V. I., Ordinary Differential Equations. 3rd ed. New York: Springer, 2006.
13. Bhatia, N. P. and G. P. Szego, Stability Theory of Dynamical Systems. New York: Springer, 2002.
14. Birkhoff, G. and G.-C. Rota, Ordinary Differential Equations. 4th ed. New York: Wiley, 1989.
15. Brauer, F. and J. A. Nohel, Qualitative Theory of Ordinary Differential Equations. Mineola, NY: Dover, 1994.
16. Churchill, R. V., Operational Mathematics. 3rd ed. New York: McGraw-Hill, 1972.
17. Coddington, E. A. and R. Carlson, Linear Ordinary Differential Equations. Philadelphia: SIAM, 1997.
18. Coddington, E. A. and N. Levinson, Theory of Ordinary Differential Equations. Malabar, FL: Krieger, 1984.
19. Dong, T.-R. et al., Qualitative Theory of Differential Equations. Providence, RI: American Mathematical Society, 1992.
20. Erdélyi, A. et al., Tables of Integral Transforms. 2 vols. New York: McGraw-Hill, 1954.
21. Hartman, P., Ordinary Differential Equations. 2nd ed. Philadelphia: SIAM, 2002.
22. Ince, E. L., Ordinary Differential Equations. New York: Dover, 1956.
23. Schiff, J. L., The Laplace Transform: Theory and Applications. New York: Springer, 1999.
24. Watson, G. N., A Treatise on the Theory of Bessel Functions. 2nd ed. Reprinted. New York: Cambridge University Press, 1995.
25. Widder, D. V., The Laplace Transform. Princeton, NJ: Princeton University Press, 1941.
26. Zwillinger, D., Handbook of Differential Equations. 3rd ed. New York: Academic Press, 1998.
27. Bellman, R., Introduction to Matrix Analysis. 2nd ed. Philadelphia: SIAM, 1997.
28. Chatelin, F., Eigenvalues of Matrices. New York: Wiley-Interscience, 1993.

29. Gantmacher, F. R., The Theory of Matrices. 2 vols. Providence, RI: American Mathematical Society, 2000.
30. Gohberg, I. P. et al., Invariant Subspaces of Matrices with Applications. New York: Wiley, 2006.
31. Greub, W. H., Linear Algebra. 4th ed. New York: Springer, 1975.
32. Herstein, I. N., Abstract Algebra. 3rd ed. New York: Wiley, 1996.
33. Joshi, A. W., Matrices and Tensors in Physics. 3rd ed. New York: Wiley, 1995.
34. Lang, S., Linear Algebra. 3rd ed. New York: Springer, 1996.
35. Nef, W., Linear Algebra. 2nd ed. New York: Dover, 1988.
36. Parlett, B., The Symmetric Eigenvalue Problem. Philadelphia: SIAM, 1998.
37. Antimirov, M. Ya., Applied Integral Transforms. Providence, RI: American Mathematical Society, 1993.
38. Bracewell, R., The Fourier Transform and Its Applications. 3rd ed. New York: McGraw-Hill, 2000.
39. Carslaw, H. S. and J. C. Jaeger, Conduction of Heat in Solids. 2nd ed. Reprinted. Oxford: Clarendon, 2000.
40. Churchill, R. V. and J. W. Brown, Fourier Series and Boundary Value Problems. 6th ed. New York: McGraw-Hill, 2006.
41. DuChateau, P. and D. Zachmann, Applied Partial Differential Equations. Mineola, NY: Dover, 2002.
42. Hanna, J. R. and J. H. Rowland, Fourier Series, Transforms, and Boundary Value Problems. 2nd ed. New York: Wiley, 2008.
43. Jerri, A. J., The Gibbs Phenomenon in Fourier Analysis, Splines, and Wavelet Approximations. Boston: Kluwer, 1998.
44. John, F., Partial Differential Equations. 4th edition New York: Springer, 1982.
45. Tolstov, G. P., Fourier Series. New York: Dover, 1976.
46. Widder, D. V., The Heat Equation. New York: Academic Press, 1975.
47. Zauderer, E., Partial Differential Equations of Applied Mathematics. 3rd ed. New York: Wiley, 2006.
48. Zygmund, A. and R. Fefferman, Trigonometric Series. 3rd ed. New York: Cambridge University Press, 2002.
49. G. B. Thomas and R. L. Finney, Calculus, 9th Ed., Pearson Education, Delhi.
50. M. J. Strauss, G. L. Bradley and K. J. Smith, Calculus, Dorling Kindersley (India) Pvt. Ltd. (Pearson Education), Delhi.
51. E. Marsden, A. J. Tromba and A. Weinstein, Basic Multivariable Calculus, Springer (SIE), Indian reprint.
52. James Stewart, Multivariable Calculus, Concepts and Contexts, Brooks /Cole, Thomson Learning, USA.
53. L. J. Goldstein, D. C. Lay and N. H. Asmar and D. I. Schneider, Calculus and its applications, Pearson.
54. Courant and John, Introduction to Calculus and Analysis, Vol II, Springer.
55. W. Rudin, Principles of Mathematical Analysis, Tata McGraw-Hill.
56. I. N. Sneddon, Elements of Partial Differential Equations, McGraw Hill.
57. L. C. Evans, Partial Differential Equations, American Mathematical Society Press.
58. P. J. Oliver, Introduction to Partial Differential Equations, Springer.

59. Tyn Myint-U, L. Debnath, Linear Partial Differential Equations for Scientists and Engineers, Springer.
60. S. L. Ross, Differential Equations, John Wiley and Sons.
61. F. H. Miller, Partial Differential Equations, John Wiley and Sons.
62. G. B. Folland, Introduction to Partial Differential Equations, Princeton University Press.